SCIREA Journal of Mathematics
http://www.scirea.org/journal/Mathematics
September 12, 2022
Volume 7, Issue 5, October 2022
https://doi.org/10.54647/mathematics11348

Feedback Functions in Problems of Nonlinear Programming

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Abstract

The article considers a method for solving a nonlinear programming problem. This method uses special feedback functions that describe the relationship between direct variables and Lagrange multipliers. These relations are similar to the conditions of the Karush-Kuhn-Tucker theorem, but do not use inequalities in their notation. With the help of feedback functions, a modified Lagrange function is constructed, the saddle points of which can be used as approximate solution of a nonlinear programming problem. The rationale for this scheme is given and an example of its use is demonstrated.

Keywords: nonlinear programming problem. Feedback functions. Modified Lagrange function. Saddle trajectory. Sequential extrapolation method.

1. Introduction

This article discusses the use of feedback functions (i.e., functions that implement, under optimality conditions, additional relationships between direct variables and Lagrange multipliers) for solving nonlinear programming problems of the form:

$$F(x) \to \max,$$
$$\textit{under conditions} \quad f_i(x) \le 0 \;\; \forall i = \overline{1,m}, \quad x_j \ge 0 \;\; \forall j = \overline{1,n}, \tag{1.1}$$

where

- $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in E^n$ – n-dimensional Euclidean space,
- functions $F(x), f_i(x) \quad \forall i = \overline{1,m}$ are defined and twice continuously differentiable in non-negative orthant in E^n.

We will also assume that problem (1.1) has local solution x^* (perhaps not unique) and denote $F(x^*)$ as F^*.

The considered method for solving problem (1.1) is based on the same ideas that were used in [1] to solve the linear programming problem. Therefore, in the next section, we first give a brief description of these ideas for the linear case, and then describe the scheme for using them in solving nonlinear problems.

2. Method of feedback functions for nonlinear problems of mathematical programming

Let the functions $F(x), f_i(x) \quad \forall i = \overline{1,m}$ be linear in E_+^n – non-negative orthant of E^n. Then problem (1.1) has the form

$$F(x) = \sum_{j=1}^{n} \sigma_j x_j \to \max \quad x \in E_+^n \qquad \textit{subject to} \qquad f_i(x) = -\beta_i + \sum_{j=1}^{n} \alpha_{ij} x_j \le 0 \quad \forall i = \overline{1,m}, \tag{2.1}$$

where $\sigma_j, \beta_i, \alpha_{ij} \quad \forall j = \overline{1,n}, \quad \forall i = \overline{1,m}$ are constants.

Dual linear problem in this case is

$$G(\lambda) = \sum_{i=1}^{m} \beta_i \lambda_i \to \min \quad \lambda \in E_+^m \quad \textit{subject to} \qquad g_j(\lambda) = -\sigma_j + \sum_{i=1}^{m} \alpha_{ij} \lambda_i \ge 0 \quad \forall j = \overline{1,n}, \tag{2.2}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)^{\mathrm{T}}$. The solution of (2.2) we will denote by $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*)^{\mathrm{T}}$.

To solve problem (2.1), we apply a special variant of the penalty function method [2,3,4]. This method consists in a sequential unconstrained search for extrema for the auxiliary function

$$A_P(\tau, x) = F(x) - \sum_{i=1}^{m} P(\tau, f_i(x)) - \sum_{j=1}^{n} P(\tau, -x_j),$$

where the penalty function $P(\tau, s)$ satisfies the following conditions:

1°. $\tau > 0 \quad \lim_{\tau \to +0} P(\tau, s) = \begin{cases} +\infty, & s > 0, \\ 0, & s < 0. \end{cases}$ and this passage to the limit is monotonic.

2°. The function $P(\tau, s)$ has continuous partial derivatives over all its arguments up to the second order.

3°. $\forall \tau > 0 \quad \forall s: \quad \dfrac{\partial P}{\partial s} > 0, \quad \dfrac{\partial^2 P}{\partial s^2} > 0.$ (2.3)

Note that, for example, the quadratic penalty function of the form $P(\tau,s)=\begin{cases} 0, & s<0, \\ \frac{s^2}{2\tau}, & s>0 \end{cases}$ does not meet these conditions.

For the dual problem, the auxiliary function (to be minimized) will have the form

$$A_D(\tau,\lambda)=G(\lambda)+\sum_{j=1}^{n}P(\tau,-g_i(\lambda))+\sum_{i=1}^{m}P(\tau,-\lambda_i)$$

Let we have: $\tilde{x}(\tau)=\arg\max_x A_P(\tau,x)$ and $\hat{\lambda}(\tau)=\arg\min_\lambda A_D(\tau,\lambda)$. Suppose further (the theory of the penalty functions method allows this option), that the following relations are valid:

$$x_j^*=\lim_{\tau\to+0}\tilde{x}_j(\tau)\quad\forall j=\overline{1,n}\qquad\text{and}\qquad\lambda_i^*=\lim_{\tau\to+0}\hat{\lambda}_i(\tau)\quad\forall i=\overline{1,m},$$

as well as

$$\lambda_i^*=\lim_{\tau\to+0}\frac{\partial P}{\partial s}(\tau,f_i(\tilde{x}(\tau)))\quad\forall i=\overline{1,m}\qquad\text{and}\qquad x_j^*=\lim_{\tau\to+0}\frac{\partial P}{\partial s}(\tau,-g_j(\hat{\lambda}(\tau)))\quad\forall j=\overline{1,n}.$$

In other words, for linear problems (2.1)−(2.2) components of vector λ^* are the Lagrange multipliers for problem (2.1), and the components of the vector x^* are the Lagrange multipliers in problem (2.2) [3]. But for a fixed $\tau>0$, generally speaking,

$$\frac{\partial P}{\partial s}(\tau,f_i(\tilde{x}(\tau)))\neq\hat{\lambda}_i(\tau)\quad\forall i=\overline{1,m}\qquad\text{and}\qquad\frac{\partial P}{\partial s}(\tau,-g_j(\hat{\lambda}(\tau)))\neq\tilde{x}_j(\tau)\quad\forall j=\overline{1,n}.$$

These ratios turn into true equalities only by passage to the limit $\tau\to+0$. However formally, we can combine them into a system of equations, which are valid $\forall\tau>0$. The system is formulated in terms of new unknowns $\bar{x}(\tau)$ and $\bar{\lambda}(\tau)$ in form of

$$\begin{cases} \bar{\lambda}_i=\frac{\partial P}{\partial s}(\tau,\ f_i(\bar{x}(\tau))) & \forall i=\overline{1,m}, \\ \bar{x}_j=\frac{\partial P}{\partial s}(\tau,-g_j(\bar{\lambda}(\tau))) & \forall j=\overline{1,n}. \end{cases}\tag{2.4}$$

In [1] is shown that (2.4) has solutions for any problems (2.1)−(2.2). For compatible problems (2.1)−(2.2)) the relation $F^*=\lim_{\tau\to+0}F(\bar{x}(\tau))$ is true. Moreover, if x^* and λ^* are unique, then there are also equalities:

$$\lambda_i^*=\lim_{\tau\to+0}\frac{\partial P}{\partial s}(\tau,f_i(\bar{x}(\tau)))\quad\forall i=\overline{1,m}\qquad\text{and}\qquad x_j^*=\lim_{\tau\to+0}\frac{\partial P}{\partial s}(\tau,-g_j(\bar{\lambda}(\tau)))\quad\forall j=\overline{1,n}.$$

Both vector functions $\bar{x}(\tau)$ and $\bar{\lambda}(\tau)$ can be used as approximate solutions to problems (2.1) and (2.2). Note that in system (2.4) there are no explicit conditions for the non-negativity of the components of $\bar{x}(\tau)$ and $\bar{\lambda}(\tau)$. This fact takes place since $\frac{\partial P}{\partial s}>0$ by definition of $P(\tau,s)$.

Now we use the fact that a continuous and strictly monotonically increasing function $\frac{\partial P}{\partial s}(\tau,s)$ for $s\in(-\infty,+\infty)$ has an *inverse* function $Q(\tau,s)$. This function $Q(\tau,s)$ is also continuous and strictly monotonically increasing for $s\in(0,+\infty)$. Then system (2.4) can be written as

$$\begin{cases} f_i(\bar{x}(\tau)) = Q(\tau, \bar{\lambda}_i(\tau)) & \forall i = \overline{1,m}, \\ -g_j(\bar{\lambda}(\tau)) = Q(\tau, \bar{x}_j(\tau)) & \forall j = \overline{1,n}. \end{cases} \tag{2.5}$$

We note that the form of the system (2.5) justifies the name of *the feedback function* for the $Q(\tau,s)$ function. Examples of feedback functions are:

$$\begin{aligned} &1)\ \ Q(\tau,s) = \tau \ln s && \text{for} \quad \frac{\partial P}{\partial s} = e^{\frac{s}{\tau}}, \\ &2)\ \ Q(\tau,s) = \frac{\tau}{2}\left(s - \frac{1}{s}\right) && \text{for} \quad \frac{\partial P}{\partial s} = \sqrt{\left(\frac{s}{\tau}\right)^2 + 1} + \frac{s}{\tau}. \end{aligned} \tag{2.6}$$

To pass to *the nonlinear case*, we introduce a new auxiliary function of the form

$$U(\tau,x,\lambda) = L(x,\lambda) - \sum_{j=1}^{n} R(\tau, x_j) + \sum_{i=1}^{m} R(\tau, \lambda_i), \tag{2.7}$$

where $L(x,\lambda) = \sum_{j=1}^{n} \sigma_j x_j - \sum_{i=1}^{m} \lambda_i \left(-\beta_i + \sum_{j=1}^{n} \alpha_{ij} x_j \right)$ is the regular Lagrange function of problem (2.1), and $R(\tau,s) = \int_{\omega(\tau)}^{s} Q(\tau,u)\, du$.

The $\omega(\tau)$ function is defined as the root of the equation $Q(\tau,\omega(\tau)) = 0$, which has a unique solution for each fixed $\tau > 0$. This is true, since the function $Q(\tau,s)$ is unbounded both from below and from above and $Q(\tau,s)$ is strictly monotonically increasing in s for $s \in (0,+\infty)$.

In this case, solutions of the system (2.5) (i.e., vectors $\bar{x}(\tau)$ and $\bar{\lambda}(\tau)$) are the *stationary points* for the function $U(\tau,x,\lambda)$ with respect to variables x and λ. Consequently, system (2.5) in terms of L and Q can be written as

$$\begin{cases} \dfrac{\partial L}{\partial x_j} = Q(\tau, \bar{x}_j(\tau)) & \forall j = \overline{1,n}, \\ \dfrac{\partial L}{\partial \lambda_i} = -Q(\tau, \bar{\lambda}_i(\tau)) & \forall i = \overline{1,m}. \end{cases} \tag{2.8}$$

The Lagrange function for problem (2.1) is

$$L(x,\lambda) = F(x) - \sum_{i=1}^{m} \lambda_i f_i(x). \tag{2.9}$$

The format (2.8)–(2.9) does not depend on whether the functions $F(x), f_i(x)$ $\forall i = \overline{1,m}$ are linear or not. Therefore, it is reasonable to try take (2.7) as the definition auxiliary function $U(\tau,x,\lambda)$ in the nonlinear case and use its stationary points as estimates for solutions to problem (1.1).

The next section of the article formulates and proves the conditions under which this approach is applicable.

3. Properties of feedback functions in nonlinear problems

The main task of this section is to find out the conditions for the applicability of system (2.8) to solve the nonlinear problem (1.1).

Let problem (1.1) have bounded F^* and x^*, where x^* may not be unique. Let also the regular Lagrange function (2.9) have a saddle point $\{x^*, \lambda^*\}$ for this problem.

Moreover, we assume that there are compact sets (with non-empty interior) $\Theta_x \subseteq E^n$, $\Theta_\lambda \subseteq E^m$, such that there is at least one pair of vectors $\{ x^* \in \operatorname{int}\Theta_x, \quad \lambda^* \in \operatorname{int}\Theta_\lambda \}$ for which $L(x^*, \lambda^*) = F^*$.

We will also use sets $\Omega_x = \Theta_x \bigcap E^n_+$, $\Omega_\lambda = \Theta_\lambda \bigcap E^m_+$. Note that under the assumptions made, the Lagrange function $L(x, \lambda)$ is continuous on the set $\Omega = \Omega_x \times \Omega_\lambda$.

We now introduce the feedback function $Q(\tau, s)$ for $\tau > 0$ and $s \in (0, +\infty)$ in the following special form: $Q(\tau, s) = \tau \Psi(s)$, where $\Psi(s)$ has the following properties

3-1°. $\Psi(s)$ strictly monotonically increases in s and has monotonic limit transitions

$$\lim_{s \to 0+0} \Psi(s) = -\infty, \qquad \lim_{s \to +\infty} \Psi(s) = +\infty .$$

3-2°. $\Psi(s)$ is continuously differentiable.

3-3°. There is a function $R(\tau, s)$ such that

$$R(\tau, s) = \tau \int_{\omega}^{s} \Psi(u)\, du , \tag{3.1}$$

where ω is a solution to the equation $\Psi(\omega) = 0$. This solution obviously exists and is unique under the assumptions made.

Such a choice of the $Q(\tau, s)$ function, on the one hand, greatly simplifies further reasoning and, on the other hand, is a particular case of the method for constructing the feedback function described in Section 2.

Indeed, let function $\frac{\partial P}{\partial s}(\tau, s)$ have the form $\Phi\left(\frac{s}{\tau}\right)$, where the function of one variable $\Phi(u)$ is such that all conditions (2.3) are satisfied. In this case, if $\Psi(u)$ is the inverse function for $\Phi(u)$, then by the definition of the inverse function we have

$$\frac{\partial P}{\partial s}(\tau, Q) = s \quad \Rightarrow \quad \Phi\left(\frac{Q}{\tau}\right) = s \quad \Rightarrow \quad \frac{Q}{\tau} = \Psi(s) \quad \Rightarrow \quad Q(\tau, s) = \tau \Psi(s) .$$

Functions (2.6) can serve as examples of such feedback functions.

Using formula (2.7) as a definition, we introduce for problem (1.1) an auxiliary function

$$U(\tau, x, \lambda) = L(x, \lambda) + W(\tau, x, \lambda), \qquad \text{where } W(\tau, x, \lambda) = -\sum_{j=1}^{n} R(\tau, x_j) + \sum_{i=1}^{m} R(\tau, \lambda_i). \tag{3.2}$$

The properties of the functions $R(\tau, s), W(\tau, x, \lambda)$ and $U(\tau, x, \lambda)$ are described by the following statements.

Lemma 3.1. 1) Function $R(\tau, s)$ is non-negative, twice continuously differentiable for any fixed $\tau > 0$ and $\forall s \in (0, +\infty)$.

For any fixed $s > 0$ $\lim_{\tau \to +0} R(\tau, s) = 0$. Moreover, this transition to the limit is monotonic.

2) The $R(\tau, s)$ function is strictly convex in s.

3) At $s = \omega$ $R(\tau, s)$ has a single minimum with $R(\tau, \omega) = 0$.

Proof. 1) Due to (3.1) and the properties of the integral, we have $R(\tau, s) \geq 0$. The $R(\tau, s)$ function is twice continuously differentiable, since the $Q(\tau, s)$ function is continuously differentiable. The equality $\lim_{\tau \to +0} R(\tau, s) = 0$ and monotonicity of this passage to the limit also follow from (3.1).

2) The $Q(\tau, s)$ function is strictly monotonically increasing in s and is continuously differentiable, hence $\frac{\partial Q}{\partial s} = \frac{\partial^2 R}{\partial s^2} > 0$ and $R(\tau, s)$ is strictly convex in s.

3) We have $R(\tau, \omega) = 0,\ \frac{\partial R}{\partial s}(\tau, \omega) = 0,\ \frac{\partial^2 R}{\partial s^2}(\tau, \omega) > 0$. Therefore, the twice continuously differentiable function $R(\tau, s)$ has a unique minimum and with a null value.

Lemma 3.2. Function $W(\tau, x, \lambda)$ has for any fixed $\tau > 0$:

1) a strict zero minimum in $\lambda \in E_+^m$ at the point with coordinates $\lambda_i = \omega \quad \forall i = \overline{1, m}$ for every fixed $x \in \Omega_x$.
2) a strict zero maximum in $x \in E_+^n$ at the point with coordinates $x_j = \omega \quad \forall j = \overline{1, n}$ for every fixed $\lambda \in \Omega_\lambda$.

Proof. 1) We take into account Lemma 3.1 and the following:

separability of the function $W(\tau, x, \lambda)$ over the components of λ,

equalities $\frac{\partial W}{\partial \lambda_i} = \left.\frac{\partial R}{\partial s}\right|_{s=\omega} = Q(\tau, \omega) = 0 \quad \forall i = \overline{1, m}$ and

ratios $\frac{\partial^2 W}{\partial \lambda_i^2} = \left.\frac{\partial^2 R}{\partial s^2}\right|_{s=\omega} > 0 \quad \forall i = \overline{1, m}$.

Based on sufficient conditions for a strict minimum, we conclude that the $W(\tau, x, \lambda)$ function has a strict global minimum in λ at the point with coordinates $\lambda_i = \omega \quad \forall i = \overline{1, m}$.

2) The proof of the second assertion is similar to the first.

By virtue of the assumptions made, the set $\Omega = \Omega_x \times \Omega_\lambda$ is a compact set in $E_+^n \times E_+^m$. Then the following theorem is true.

Theorem 3.1. For fixed $\tau > 0$, the function $U(\tau, x, \lambda)$ has a saddle point $\{\bar{x}, \bar{\lambda}\}$ in the set Ω, where $\bar{x}$ and $\bar{\lambda}$ are solutions of the system of equations

$$\begin{cases} \operatorname{grad}_x U(\tau,\bar{x},\bar{\lambda}) = o, \\ \operatorname{grad}_\lambda U(\tau,\bar{x},\bar{\lambda}) = o. \end{cases} \tag{3.3}$$

Proof.

1) Let us first show that on the set Ω the function $U(\tau,x,\lambda)$ has a maximum in x for each fixed $\lambda \in \Omega_\lambda$ and a minimum in λ for each fixed $x \in \Omega_x$.

   Under the assumptions made and by virtue of Lemma 3.2, both the Lagrange function and the $W(\tau,x,\lambda)$ function will have this property. According to (3.2) $U(\tau,x,\lambda) = L(x,\lambda) + W(\tau,x,\lambda)$ on the set Ω, which is a compact set. Therefore, the $U(\tau,x,\lambda)$ function will also have this property.

2) It follows from the theorems (for example, on *a saddle point* [5,6,7,8]) that in the case under consideration the function $U(\tau,x,\lambda)$ has a saddle point on the compact Ω.

3) According to Lemma 3.1, the $R(\tau,s)$ function has a minimum at $s=\omega$, where $\omega \in (0,+\infty)$. Therefore, the saddle point of the $U(\tau,x,\lambda)$ function belongs to the *interior* of the set Ω. Namely, due to (3.1), the definitions of the sets Θ_x and Θ_λ, the strict convexity of $R(\tau,s)$, and the relations $\lim_{s\to 0+0} R(\tau,s) = \lim_{s\to+\infty} R(\tau,s) = +\infty$, valid for any fixed $\tau > 0$.

4) Finally, from the fact that the function $U(\tau,x,\lambda)$ is twice continuously differentiable it follows that this saddle point is stationary and, therefore, is determined by the system of equations (3.3). Thus we come to the conclusion that the assertion of the theorem is true.

Note that system (3.3) can also be written in the form

$$\begin{cases} \dfrac{\partial L}{\partial x_j} = Q(\tau,\bar{x}_j) \quad \forall j=\overline{1,n}, \\ \dfrac{\partial L}{\partial \lambda_i} = -Q(\tau,\bar{\lambda}_i) \quad \forall i=\overline{1,m}. \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{\partial F}{\partial x_j}(\bar{x}) - \sum_{i=1}^{m} \bar{\lambda}_i \dfrac{\partial f_i}{\partial x_j}(\bar{x}) = Q(\tau,\bar{x}_j) \quad \forall j=\overline{1,n}, \\ f_i(\bar{x}) = Q(\tau,\bar{\lambda}_i) \quad \forall i=\overline{1,m}. \end{cases} \tag{3.4}$$

The following remarks may also be made here.

1. The system (3.4) justifies for $Q(\tau,s)$ naming *feedback function* in nonlinear case.
2. System (3.4) is similar to the conditions of the Karush-Kuhn-Tucker theorem, but does not explicitly contain conditions for the non-negativity of the Lagrange multipliers. These conditions are provided by the properties of the $Q(\tau,s)$ function.
3. The $U(\tau,x,\lambda)$ function can also be viewed as the result of a special modification Lagrange functions. Different kinds such modifications are described in a large number of monographs and articles, for example, in [9].
4. With the help of (3.4) it is easy to verify the validity of the conditions of *complementary slackness*. Indeed, we have

   $$\bar{\lambda}_i f_i(\bar{x}) = \bar{\lambda}_i Q(\tau,\bar{\lambda}_i) = \tau \bar{\lambda}_i \Psi(\bar{\lambda}_i) \to 0 \quad \forall i=\overline{1,m}, \quad \text{when } \tau \to +0.$$

Theorem 3.2. 1) For the auxiliary function $U(\tau, x, \lambda)$ equality

$$\lim_{\tau \to +0} U(\tau, x, \lambda) = L(x, \lambda) \quad \forall \{x, \lambda\} \in \Omega$$

takes place.

2) Under the assumptions made above, it will be true

$$\lim_{\tau \to +0} U(\tau, \bar{x}(\tau), \bar{\lambda}(\tau)) = F^* . \tag{3.5}$$

3) In the case of local uniqueness of the solution of problem (1.1), the equality

$$\lim_{\tau \to +0} \bar{x}(\tau) = x^* \tag{3.6}$$

will be true.

Proof.

1) Let's estimate the difference between the values of the auxiliary function U and the Lagrage function L. For the point $\{x, \lambda\} \in \Omega$, by virtue of Lemma 3.1 and (3.2) we have

$$U(\tau, x, \lambda) = L(x, \lambda) - \sum_{j=1}^{n} R(\tau, x_j) + \sum_{i=1}^{m} R(\tau, \lambda_i) \geq L(x, \lambda) - \sum_{j=1}^{n} R(\tau, x_j) \geq L(x, \lambda) - n \max_{\left\{ \substack{x \in \Omega_x \\ j = \overline{1,n}} \right\}} R(\tau, x_j) =$$

$$L(x, \lambda) + n \min R(\tau, x_j).$$

Similarly

$$U(\tau, x, \lambda) = L(x, \lambda) - \sum_{j=1}^{n} R(\tau, x_j) + \sum_{i=1}^{m} R(\tau, \lambda_i) \leq L(x, \lambda) + \sum_{i=1}^{m} R(\tau, \lambda_i) \leq L(x, \lambda) + m \max_{\left\{ \substack{\lambda \in \Omega_\lambda \\ i = \overline{1,m}} \right\}} R(\tau, \lambda_i) =$$

$$L(x, \lambda) - m \min_{\left\{ \substack{\lambda \in \Omega_\lambda \\ i = \overline{1,m}} \right\}} R(\tau, \lambda_i).$$

As shown earlier, for $\tau > 0$ and $s \in (0, +\infty)$ we have

$$R(\tau, s) \geq 0, \quad \lim_{\tau \to +0} R(\tau, s) = 0, \quad \min_{s \in (0, +\infty)} R(\tau, s) = 0 .$$

Then, for example, according to Lemma 5.1 [6], the relations

$$\lim_{\tau \to +0} \min_{s \in (0, +\infty)} R(\tau, s) = \min_{s \in (0, +\infty)} \lim_{\tau \to +0} R(\tau, s) = \min_{s \in (0, +\infty)} R(\tau, s) = 0$$

are valid. This gives

$$\begin{cases} \lim\limits_{\tau \to +0} \min\limits_{\left\{ \substack{x \in \Omega_x \\ j = \overline{1,n}} \right\}} R(\tau, x_j) = \min\limits_{\left\{ \substack{x \in \Omega_x \\ j = \overline{1,n}} \right\}} \lim\limits_{\tau \to +0} R(\tau, x_j) = 0, \\ \lim\limits_{\tau \to +0} \min\limits_{\left\{ \substack{\lambda \in \Omega_{\lambda x} \\ i = \overline{1,m}} \right\}} R(\tau, \lambda_i) = \min\limits_{\left\{ \substack{\lambda \in \Omega_{\lambda x} \\ i = \overline{1,m}} \right\}} \lim\limits_{\tau \to +0} R(\tau, \lambda_i) = 0. \end{cases} \tag{3.7}$$

Thus we get estimates $L(x, \lambda) + n \min\limits_{\left\{ \substack{x \in \Omega_x \\ j = \overline{1,n}} \right\}} R(\tau, x_j) \leq U(\tau, x, \lambda) \leq L(x, \lambda) - m \min\limits_{\left\{ \substack{\lambda \in \Omega_\lambda \\ i = \overline{1,m}} \right\}} R(\tau, \lambda_i)$.

Then, using (3.7), we come to the conclusion that $\lim\limits_{\tau \to +0} U(\tau, x, \lambda) = L(x, \lambda) \quad \forall \{x, \lambda\} \in \Omega$.

2) Let $\tau_0 > 0$ be such that $\forall \tau \in (0, \tau_0] \quad \begin{cases} \bar{x}(\tau) \in \Omega_x, \\ \bar{\lambda}(\tau) \in \Omega_\lambda. \end{cases}$ Then by the assumptions made, we have $\max\limits_{x \in \Omega_x} \min\limits_{\lambda \in \Omega_\lambda} L(x, \lambda) = F^*$. On the other hand, arguing as in 1), we obtain

$$\max_{x\in\Omega_x}\min_{\lambda\in\Omega_\lambda} L(x,\lambda)=\max_{x\in\Omega_x}\min_{\lambda\in\Omega_\lambda}\lim_{\tau\to+0} U(\tau,x,\lambda)=\lim_{\tau\to+0}\max_{x\in\Omega_x}\min_{\lambda\in\Omega_\lambda} U(\tau,x,\lambda)=\lim_{\tau\to+0} U\left(\tau,\bar{x}(\tau),\bar{\lambda}(\tau)\right).$$

This gives $\lim_{\tau\to+0} U\left(\tau,\bar{x}(\tau),\bar{\lambda}(\tau)\right)=F^*$.

3) $\{\bar{x}(\tau),\bar{\lambda}(\tau)\}$ belongs to Ω, which is bounded as a compact set. Then by theorem Bolzano-Weierstrass, $\{\bar{x}(\tau),\bar{\lambda}(\tau)\}$ has a limit point $\{\bar{\bar{x}},\bar{\bar{\lambda}}\}$ at $\tau\to+0$ for which $L(\bar{\bar{x}},\bar{\bar{\lambda}})=F^*=L(x^*,\lambda^*)$. Due to the continuity of the Lagrange function $L(x,\lambda)$ if the solution to problem (1.1) is unique, then $x^*=\bar{\bar{x}}=\lim_{\tau\to+0}\bar{x}(\tau)$.

Note also that Theorem 3.2 implies the following conclusion.

> Let some necessary (and/or sufficient) local optimality condition for problem (1.1) formulated in terms of the Lagrange function *L*. Then this condition can also be formulated in terms of the auxiliary function U.

Let us call the vector function $\left\{\bar{x}(\tau),\bar{\lambda}(\tau)\right\}$ $\forall\,\tau\in(0,\tau_0)$ the *saddle trajectory* of problem (1.1). An important property of the vector function $\left\{\bar{x}(\tau),\bar{\lambda}(\tau)\right\}$ is described by the following theorem.

Theorem 3.3. On the saddle trajectory vector functions $\left\{\bar{x}(\tau),\bar{\lambda}(\tau)\right\}$ are continuously differentiable $\forall\,\tau\in(0,\tau_0)$.

Proof.

1) Let us first show that the determinant of the Jacobian matrix system (3.3) is not equal to zero. This matrix has the form

$$H=\left(\begin{array}{c|c}\dfrac{\partial^2 U}{\partial x_j\partial x_k}\ \forall j,k=\overline{1,n} & \dfrac{\partial^2 U}{\partial x_j\partial\lambda_q}\ \forall j=\overline{1,n},\ \forall q=\overline{1,m}\\ \hline \dfrac{\partial^2 U}{\partial\lambda_i\partial x_k}\ \forall i=\overline{1,m},\ \forall k=\overline{1,n} & \dfrac{\partial^2 U}{\partial\lambda_i\partial\lambda_q}\ \forall i,q=\overline{1,m}\end{array}\right).$$

By virtue of equalities (2.9), (3.1) and (3.2) we have

$$\frac{\partial^2 U}{\partial x_j\partial x_k}=A_{jk}=\frac{\partial^2 L}{\partial x_j\partial x_k}-\delta_{jk}\frac{\partial Q}{\partial s}(x_k),\qquad \frac{\partial^2 U}{\partial x_j\partial\lambda_q}=B_{jq}=-\frac{\partial f_q}{\partial x_j},$$

$$\frac{\partial^2 U}{\partial\lambda_i\partial\lambda_q}=C_{iq}=\delta_{iq}\frac{\partial Q}{\partial s}(\lambda_q),$$

where δ_{ij} is the Kronecker symbol, i.e. $\delta_{ij}=\begin{cases}1,\text{ if } i=j,\\ 0,\text{ if } i\neq j.\end{cases}$ Or, in block form, $H=\left(\begin{array}{c|c}A & B\\ B^{\mathrm{T}} & C\end{array}\right)$.

2) If $\tau\in(0,\tau_0]$, then the determinant *A* is non-zero.

Indeed, the square matrix A is the sum of the $\nabla^2 L$ matrix and diagonal matrix with elements $-\delta_{jk}\dfrac{\partial Q}{\partial s}(x_k)$. The first one is negative semi-definite by the assumption about properties of the Lagrange function in Ω. The second matrix is strictly negative definite, because

$\frac{\partial Q}{\partial s} = \frac{\partial^2 R}{\partial s^2} > 0$ (Lemma 3.1). Then the total matrix A will be strictly negative definite and by the Sylvester criterion $\det A \neq 0$. In this case, A^{-1} also exists.

Similar reasoning gives: the matrix C is postive definite and $\det C > 0$.

3) We use the Schur formula $\det H = \det A \cdot \det\left(C - B^{\mathrm{T}} A^{-1} B\right)$. From the Binet-Cauchy theorem [10] we have that for any rank B, the matrix $B^{\mathrm{T}} A^{-1} B$ is either negative definite or negative semidefinite. Then the matrix $C - B^{\mathrm{T}} A^{-1} B$ is positive definite and $\det\left(C - B^{\mathrm{T}} A^{-1} B\right) \neq 0$.

   Finally, according to the Schur formula $\det H = \det A \cdot \det\left(C - B^{\mathrm{T}} A^{-1} B\right) \neq 0$.

4) Let $\tau \in (0, \tau_0]$. Consider the system of equations (3.3) as an implicitly specifying vector function $\{\bar{x}(\tau), \bar{\lambda}(\tau)\}$. For fixed $\tau \in (0, \tau_0]$ we have:

   1. Left sides of (3.3) are continuously differentiable functions.
   2. Jacobi matrix determinant for system (3.3) (or, which is the same, the Hessian of the auxiliary function U) is not equal to zero.

   Then the implicit function theorem [11, 12] is applied for the system. The theorem implies the existence of a continuously differentiable vector function $\{\bar{x}(\tau), \bar{\lambda}(\tau)\}$ for fixed $\tau \in (0, \tau_0]$.

4. Sequential linear extrapolation for feedback functions

According the theorem 3.3 vector-functions $\{\bar{x}(\tau), \bar{\lambda}(\tau)\}$ are continuous differentiable in τ in the saddle trajectory for problem (1.1). It makes possible to estimate x^* and λ^* without direct solving (3.4) for small $\tau \in (0, \tau_0]$. The suggested method is based on the Taylor approximations of $\{\bar{x}(\tau), \bar{\lambda}(\tau)\}$.

Let $\tau \in (0, \tau_0]$. For small enough $\Delta > 0$ these approximations are

$$\bar{x}(\tau + \Delta) = \bar{x}(\tau) + \frac{d\bar{x}}{d\tau}\Delta + o(\Delta) \quad \text{and} \quad \bar{\lambda}(\tau + \Delta) = \bar{\lambda}(\tau) + \frac{d\bar{\lambda}}{d\tau}\Delta + o(\Delta).$$

Passing to the limit with $\Delta \to -\tau + 0$, we get the following estimates for x^* and λ^*:

$$x^+ = \bar{x}(\tau) - \frac{d\bar{x}}{d\tau}\tau \quad \text{and} \quad \lambda^+ = \bar{\lambda}(\tau) - \frac{d\bar{\lambda}}{d\tau}\tau. \tag{4.1}$$

Values for the components of the derivatives in (4.1) are found (according to the implicit function theorem) from the system of linear equations

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \frac{\partial^2 U}{\partial x_p \partial x_j} \cdot \frac{d\bar{x}_j}{d\tau} + \sum_{i=1}^{m} \frac{\partial^2 U}{\partial x_p \partial \lambda_i} \cdot \frac{d\bar{\lambda}_i}{d\tau} = -\frac{\partial^2 U}{\partial x_p \partial \tau} & \forall p = \overline{1,n}, \\ \displaystyle\sum_{j=1}^{n} \frac{\partial^2 U}{\partial \lambda_q \partial x_j} \cdot \frac{d\bar{x}_j}{d\tau} + \sum_{i=1}^{m} \frac{\partial^2 U}{\partial \lambda_q \partial \lambda_i} \cdot \frac{d\bar{\lambda}_i}{d\tau} = -\frac{\partial^2 U}{\partial \lambda_q \partial \tau} & \forall q = \overline{1,m}. \end{cases} \tag{4.2}$$

Equations (4.2) are obtained by differentiating the equations of system (3.3) with respect to the variable τ.

Let the coefficient $\tau > 0$ is small enough in absolute value. By properties of Taylor's approximation the norm of the difference between of points $\{x^{+}, \lambda^{+}\}$ and $\{x^{*}, \lambda^{*}\}$ will be less than the norm of the difference of $\{\bar{x}(\tau), \bar{\lambda}(\tau)\}$ and $\{x^{*}, \lambda^{*}\}$.

If the achieved accuracy is not sufficient, we may repeat calculations (4.1)−(4.2) at point $\{x^{+}, \lambda^{+}\}$ subject to the following modification.

Direct reusing formulas (4.1)−(4.2) is not possible. Indeed, the point $\{x^{+}, \lambda^{+}\}$ generally does not belong to the saddle trajectory.

To overcome this difficulty let us change the function $W(\tau, x, \lambda)$ so that the point $\{x^{+}, \lambda^{+}\}$ belongs to some new saddle trajectory. This new trajectory must have the same properties as the original one. Fig.1 illustrates this idea.

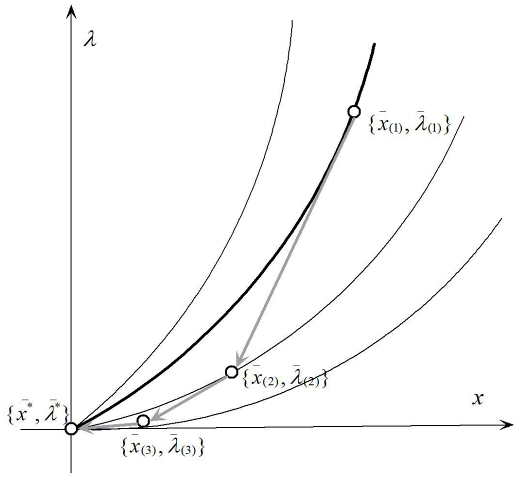

Fig. . Saddle trajectories in sequential linear extrapolation.

The suggested modification consists of the following.

We replace the scalar parameter τ in the function $W(\tau, x, \lambda)$ with $(n+m)$-component vector $\vec{\tau}$. This vector has coordinates $\tau_{(t)xj}$ $\forall j = \overline{1,n}$ and $\tau_{(t)\lambda i}$ $\forall i = \overline{1,m}$, where $t = 1,2,3,\ldots$ is number of extrapolating step.

All $\tau_{(t)xj}$ $\forall j = \overline{1,n}$ and $\tau_{(t)\lambda i}$ $\forall i = \overline{1,m}$l are introduced similarly to τ in §3. Then the main properties of

$$W\left(\vec{\tau}, x, \lambda\right) = -\sum_{j=1}^{n} R\left(\tau_{xj}, x_j\right) + \sum_{i=1}^{m} R\left(\tau_{\lambda i}, \lambda_i\right).$$

are the same as indicated in Lemma 3.2. The auxiliary function $U\left(\vec{\tau}, x, \lambda\right)$ is defined as above by (3.2).

Let $\{\bar{x}_{(t)}, \bar{\lambda}_{(t)}\}$ is the current approximation at step $t = 1,2,3,\ldots$ with vector $\vec{\tau}_{(t)}$ The approximation at next step will be defined by

$$\begin{cases} \bar{x}_{(t+1)j} = \bar{x}_{(t)j} - \dfrac{d\bar{x}_j}{d\tau_{(t)xj}} \tau_{(t)xj} & j = \overline{1,n}, \\ \bar{\lambda}_{(t+1)i} = \bar{\lambda}_{(t)i} - \dfrac{d\bar{\lambda}_i}{d\tau_{(t)xi}} \tau_{(t)xi} & i = \overline{1,m}. \end{cases} \tag{4.3}$$

The values of derivatives in (4.3) can be found using the next system of linear equations.

$$\begin{cases} \sum_{j=1}^{n} \dfrac{\partial^2 U}{\partial x_p \partial x_j} \cdot \dfrac{d\bar{x}_j}{d\tau_{(t)xj}} + \sum_{i=1}^{m} \dfrac{\partial^2 U}{\partial x_p \partial \lambda_i} \cdot \dfrac{d\bar{\lambda}_i}{d\tau_{(t)\lambda i}} = -\dfrac{\partial^2 U}{\partial x_p \partial \tau_{(t)xp}} & \forall p = \overline{1,n}, \\ \sum_{j=1}^{n} \dfrac{\partial^2 U}{\partial \lambda_q \partial x_j} \cdot \dfrac{d\bar{x}_j}{d\tau_{(t)xj}} + \sum_{i=1}^{m} \dfrac{\partial^2 U}{\partial \lambda_q \partial \lambda_i} \cdot \dfrac{d\bar{\lambda}_i}{d\tau_{(t)\lambda i}} = -\dfrac{\partial^2 U}{\partial \lambda_q \partial \tau_{(t)\lambda q}} & \forall q = \overline{1,m}. \end{cases} \tag{4.4}$$

All partial derivativesare calculated at $\{\bar{x}_{(t)}, \bar{\lambda}_{(t)}\}$ and $\vec{\tau}_{(t)}$.

Finally, components of $\vec{\tau}_{(t+1)}$ may be found from

$$\begin{cases} \operatorname*{grad}_x U\left(\vec{\tau}_{(t+1)}, \bar{x}_{(t+1)}, \bar{x}_{(t+1)}\right) = o, \\ \operatorname*{grad}_x U\left(\vec{\tau}_{(t+1)}, \bar{x}_{(t+1)}, \bar{x}_{(t+1)}\right) = o. \end{cases} \tag{4.5}$$

When using (3.1) and (3.2), system (4.5) takes the form

$$\begin{cases} \dfrac{\partial L}{\partial x_j}\left(\bar{x}_{(t)}, \bar{\lambda}_{(t)}\right) - \tau_{(t)xj}\Psi\left(\bar{x}_{(t)j}\right) = 0 \qquad j = \overline{1,n}, \\ \dfrac{\partial L}{\partial \lambda_i}\left(\bar{x}_{(t)}, \bar{\lambda}_{(t)}\right) - \tau_{(t)\lambda i}\Psi\left(\bar{\lambda}_{(t)i}\right) = 0 \qquad i = \overline{1,m} \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{\partial L}{\partial x_j}\left(\bar{x}_{(t)}, \bar{\lambda}_{(t)}\right) - \tau_{(t)xj}\Psi\left(\bar{x}_{(t)j}\right) = 0 \qquad j = \overline{1,n}, \\ f_i\left(\bar{x}_{(t)}\right) - \tau_{(t)\lambda i}\Psi\left(\bar{\lambda}_{(t)i}\right) = 0 \qquad i = \overline{1,m}. \end{cases}$$

Note that the last system is a system of $n + m$ linear equations, each of which has one unknown only. The *local* convergence of the described procedure follows from contractibility of the operator defined by formulas (4.3)−(4.4)−(4.5). Indeed, $\lim_{t\to\infty}\left\|\vec{\tau}_{(t)}\right\| = 0$ due to properties of the Taylor approximation.

Now let us illustrate the practical use of the method of feedback functions together with the linear sequential extrapolation procedure to solve the following nonlinear programming problem.

Problem 1. *Maximize in* E^2 $\quad F(x) = -(x_1 - 1)^2 - x_2^2$

under the conditions $x_1 \geq 0, x_2 \geq 0$ *and* $x_1 + 2x_2 \leq 3, \quad x_1^2 - x_2 \leq 0.$

Solution.

Let the feedback function be $Q(\tau, s) = \dfrac{\tau}{2}\left(s - \dfrac{1}{s}\right)$. Note that in this case

$$R(\tau, s) = \frac{\tau}{2}\left(\frac{s^2}{2} - \ln s - \frac{1}{2}\right) \quad \text{and} \quad \frac{\partial P}{\partial s} = \sqrt{\left(\frac{s}{\tau}\right)^2 + 1} + \frac{s}{\tau}.$$

We introduce the notations

$$\begin{aligned} f_1(x) &= -3 + x_1 + 2x_2, \\ f_2(x) &= \ x_1^2 - x_2, \\ L(x, \lambda) &= -(x_1 - 1)^2 - x_2^2 - \lambda_1(-3 + x_1 + 2x_2) - \lambda_2\left(x_1^2 - x_2\right). \end{aligned}$$

In the procedure of sequential linear extrapolation instead of τ we use four-component vector $\vec{\tau}$ with coordinate column $\left(\tau_{x1}, \tau_{x2}, \tau_{\lambda 1}, \tau_{\lambda 2}\right)^{\mathrm{T}}$. Auxiliary U function in the problem 1 has the form

$$\begin{aligned} &U(\tau, x, \lambda) = \\ &\quad -(x_1 - 1)^2 - x_2^2 - \lambda_1(-3 + x_1 + 2x_2) - \lambda_2\left(x_1^2 - x_2\right) - R\left(\tau_{x1}, x_1\right) - R\left(\tau_{x2}, x_2\right) + R\left(\tau_{\lambda 1}, \lambda_1\right) + R\left(\tau_{\lambda 2}, \lambda_2\right). \end{aligned}$$

Then the system of equations (3.4) (stationarity conditions for the *U*-function) for will be

$$
\begin{cases}
-3+\bar{x}_1+2\bar{x}_2=\dfrac{\tau_{\lambda 1}}{2}\left(\bar{\lambda}_1-\dfrac{1}{\bar{\lambda}_1}\right), \\
\bar{x}_1^2-\bar{x}_{22}=\dfrac{\tau_{\lambda 2}}{2}\left(\bar{\lambda}_2-\dfrac{1}{\bar{\lambda}_2}\right), \\
-2(\bar{x}_1-1)-\bar{\lambda}_1-2\bar{x}_1\bar{\lambda}_2=\dfrac{\tau_{x1}}{2}\left(\bar{x}_1-\dfrac{1}{\bar{x}_1}\right), \\
-2\bar{x}_2-\overline{2\lambda_1}+\bar{\lambda}_2=\dfrac{\tau_{x2}}{2}\left(\bar{x}_2-\dfrac{1}{\bar{x}_2}\right).
\end{cases}
\tag{4.6}
$$

For the initial approximation in the procedure of sequantial linear extrapolation let us take the solution of system (4.6) with $\tau=0.01$. This approximation is given in Tab.1A and Tab. 1B.

Table 1A. Solution to system (4.6) for $\tau=0.01$.

$\bar{x}_1(\tau)$	$\bar{x}_2(\tau)$	$\bar{\lambda}_1(\tau)$	$\bar{\lambda}_2(\tau)$	$F(\bar{x}(\tau))$
0.59002481	0.35181724	$2.930\cdot 10^{-3}$	0.69704208	-0.51620893

Table 1B. Solution to system (4.6) for $\tau=0.01$.

$f_1(\bar{x}(\tau))$	$f_2(\bar{x}(\tau))$	$L(\bar{x}(\tau),\bar{\lambda}(\tau))$	$U(\bar{x}(\tau),\bar{\lambda}(\tau))$
1.70634071	$-3.688\cdot 10^{3}$	0.50863832	0.48549662

The matrix of the system (4.5) is

$$
\begin{pmatrix}
-\dfrac{\tau}{2}\left(1+\dfrac{1}{\bar{x}_1^2}\right)-2-2\bar{\lambda}_2 & 0 & -1 & -2\bar{x}_1 \\
0 & -\dfrac{\tau}{2}\left(1+\dfrac{1}{\bar{x}_1^2}\right)-2 & -2 & 1 \\
-1 & -2 & \dfrac{\tau}{2}\left(1+\dfrac{1}{\bar{\lambda}_1^2}\right) & 0 \\
-2\bar{x}_1 & 1 & 0 & \dfrac{\tau}{2}\left(1+\dfrac{1}{\bar{\lambda}_2^2}\right)
\end{pmatrix},
$$

while the RHS-column of this system is

$$
\begin{pmatrix}
-\dfrac{\tau}{2}\left(\bar{x}_1-\dfrac{1}{\bar{x}_1}\right) & -\dfrac{\tau}{2}\left(\bar{x}_2-\dfrac{1}{\bar{x}_2}\right) & \dfrac{\tau}{2}\left(\bar{\lambda}_1-\dfrac{1}{\bar{\lambda}_1}\right) & \dfrac{\tau}{2}\left(\bar{\lambda}_2-\dfrac{1}{\bar{\lambda}_2}\right)
\end{pmatrix}^{\mathrm{T}}.
$$

Let the components of $\bar{x}$ and $\bar{\lambda}$ in iteration with number t $\forall t=1,2,3,\ldots$ are $\bar{x}_{1(t)},\bar{x}_{2(t)}$ and $\bar{\lambda}_{1(t)},\bar{\lambda}_{2(t)}$. Then the values of the components $\vec{\tau}_{(t+1)}$, guaranteeing that the point $\{\bar{x}_{(t+1)},\bar{\lambda}_{(t+1)}\}$ belong to some other saddle trajectory, are due to (4.4)

$$
\begin{cases}
\tau_{x1(t+1)} = \dfrac{2\left(-2\left(\bar{x}_{1(t)}-1\right)\right)-\bar{\lambda}_{1(t)}-2\bar{x}_{1(t)}\bar{\lambda}_{1(t)}}{\bar{x}_{1(t)}-\dfrac{1}{\bar{x}_{1(t)}}}, & \tau_{x2(t+1)} = \dfrac{2\left(-2\bar{x}_{1(t)}-2\bar{\lambda}_{1(t)}+\bar{\lambda}_{2(t)}\right)}{\bar{x}_{2(t)}-\dfrac{1}{\bar{x}_{2(t)}}}, \\
\tau_{\lambda1(t+1)} = \dfrac{2\left(-3+\bar{x}_{1(t)}-2\bar{x}_{2(t)}\right)}{\bar{\lambda}_{1(t)}-\dfrac{1}{\bar{\lambda}_{1(t)}}}, & \tau_{\lambda2(t+1)} = \dfrac{2\left(\bar{x}_{1(t)}^{2}-\bar{x}_{2(t)}\right)}{\bar{\lambda}_{2(t)}-\dfrac{1}{\bar{\lambda}_{2(t)}}}.
\end{cases}
$$

Value of error may be estimated at each iteration in two diffenert ways.

1) using that $\lambda_1^* = 0$ and λ_2^* is the single positive root of equation $\lambda(\lambda+1)^2 - 2 = 0$. According to the Cardano formula we have $\lambda_2^* = \dfrac{\left(\sqrt[3]{28+3\sqrt{87}}-1\right)^2}{3\sqrt[3]{28+3\sqrt{87}}} \approx 0.69562077$.
2) we can get this estimate by comparing $\left| L-F \right|$ with zero.

The results of calculations performed are given in Tab.~2, 3 and 4.

Table 2. Problem 1 (linear extrapolation).

t	$\bar{x}_{1(t)}(\tau)$	$\bar{x}_{2(t)}(\tau)$	$F\left(\bar{x}_{(t)}(\tau)\right)$	$\bar{\lambda}_{1(t)}(\tau)$	$\bar{\lambda}_{2(t)}(\tau)$
1	0.590024813	0.351817238	-0.291855023	0.002930222	0.697042081
2	0.589788382	0.347873253	-0.289289372	$-1.390\cdot10^{-5}$	0.695539663
3	0.589754522	0.347810387	-0.289289372	$1.294\cdot10^{-8}$	0.695620736
4	0.589754512	0.347810385	-0.289273424	0	0.695620770

Table 3. Problem 1 (linear extrapolation).

t	$f_1\left(\bar{x}_{(t)}(\tau)\right)$	$f_2\left(\bar{x}_{(t)}(\tau)\right)$	L	L-F	U
1	-1.706340711	-0.003687958	-0.284284404	0.007570619	-0.261142713
2	-1.714465112	$-2.291\cdot10^{-5}$	-0.289297265	$-7.893\cdot10^{-6}$	-0.289593289
3	-1.714624703	$8.824\cdot10^{-9}$	-0.289273422	$-3.920\cdot10^{-9}$	-0.289273422
4	-1.714624718	$-1.11\cdot10^{-15}$	-0.289273424	0	-0.290232652

Table 4. Problem 1 (linear extrapolation).

t	$\tau_{x1(t)}$	$\tau_{x2(t)}$	$\tau_{\lambda1(t)}$	$\tau_{\lambda2(t)}$	$\bar{\lambda}_{2(t)}-\lambda_2^*$
1	0.01	0.01	0.01	0.01	0.001421311
2	$9.565\cdot10^{-6}$	0.000141717	$-4.767\cdot10^{-5}$	$6.176\cdot10^{-5}$	$-8.111\cdot10^{-5}$
3	$-8.794\cdot10^{-9}$	$3.265\cdot10^{-8}$	$4.437\cdot10^{-9}$	$-2.379\cdot10^{-8}$	$-3.373\cdot10^{-8}$
4	$-1.41\cdot10^{-15}$	0	0	$2.99\cdot10^{-15}$	$-4.40\cdot10^{-10}$

5. A note about the practical use of the feedback function method

According to the authors, the effectiveness of the described method is the subject of a separate study. However, some conclusions can already be drawn here.

In practice, it often happens that some trial points of the numerical procedure for solving system (3.4) turn out to be outside the positive orthant. The need for special control over the fulfillment of the condition $s > 0$ can be avoided in the following way.

Let us replace the unknown $\bar{x}_j \ \forall j = \overline{1,n}$ and $\bar{\lambda}_i \ \forall i = \overline{1,m}$ in system (3.4) by their absolute values $|\bar{x}_j| \ \forall j = \overline{1,n}$ and $|\bar{\lambda}_i| \ \forall i = \overline{1,m}$. In other words, instead of (3.4) we will solve the system

$$\begin{cases} \dfrac{\partial L}{\partial \lambda_i}\left(|\bar{x}_1|, |\bar{x}_2|, \ldots, |\bar{x}_n|, |\bar{\lambda}_1|, |\bar{\lambda}_2|, \ldots, |\bar{\lambda}_m| \right) = -Q\left(\tau, |\bar{\lambda}_i|\right) & \forall i = \overline{1,m}, \\ \dfrac{\partial L}{\partial x_j}\left(|\bar{x}_1|, |\bar{x}_2|, \ldots, |\bar{x}_n|, |\bar{\lambda}_1|, |\bar{\lambda}_2|, \ldots, |\bar{\lambda}_m| \right) = Q\left(\tau, |\bar{x}_j|\right) & \forall j = \overline{1,n} \end{cases} \tag{5.1}$$

without any additional restrictions.

System solutions (5.1) may not belong to the positive orthant of the space $E^n \times E^m$. However, it is obvious that their absolute values are positive solutions of system (3.4).

In addition, such a replacement reduces the dependence of the computational procedure on the choice of the starting point.

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