SCIREA Journal of Mathematics
http://www.scirea.org/journal/Mathematics
December 11, 2022
Volume 7, Issue 6, December 2022
https://doi.org/10.54647/mathematics11371

The Minimal Norm Least Squares Solutions for a Class of Matrix Equations

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Abstract

In this paper, the minimal norm least squares solution of matrix equations $(AXC, BXD, AXD, BXC) = (E, F, G, H)$ is discussed, by using the projection theorem, the generalized singular value decomposition and the canonical correlation decomposition, the expression of the solution of this problem is obtained.

Keywords: Minium-norm least-square solution; the Generalized Singular Value Decomposition; the Canonical Correlation Decomposition; the Projection Theorem

1. The introduction

Linear matrix equation problems have important applications in biology, electricity, photon spectroscopy, vibration theory, finite element, structural design, parameter identification, automatic control theory, linear optimal control and many other fields. The solution of linear matrix equations has become one of the hot research topics in the field of computational mathematics. The literature [1] gives the least squares solution of the matrix equations, but because the form of the solution does not satisfy the orthogonal invariance of the norm, it can't directly solve the minimal norm solution. This paper successfully solves the problem

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Research Supported by the Scientific Research of Hunan Provincial Education Department ,(Grant No. 20C0178)

through the projection theorem.

Throughout the paper, we denote the set of all real $m \times n$ matrices by $R^{m\times n}$, the set of all real orthogonal matrices by $OR^{n\times n}$, the identity matrix of order n by I_n, the ith column of the identity matrix of order n by e_i, the identity matrix of order n by S_n, here $S_n = \{e_n, e_{n-1}, \cdots, e_1\}$, the transpose of the matrix by A^T, the Frobenius norm of a matrix by $\|\cdot\|$.

Generalized Singular Value Decomposition(GSVD[2] **)** Let $A_1 \in R^{m\times(n-k)}$, $A_2 \in R^{m\times k}$, $B_1 \in R^{p\times(n-k)}$, $B_2 \in R^{p\times k}$, Then, the GSVD of matrix pair (A_1, A_2), $\begin{pmatrix} B_1^T \\ B_2^T \end{pmatrix}$ respectively are

$$A_1 = M\Sigma_{A_1}U^T,\ A_2 = M\Sigma_{A_2}V^T;\quad B_1^T = P\Sigma_{B_1}N,\quad B_2^T = Q\Sigma_{B_2}N, \tag{1}$$

where, $M \in R^{m\times m}$, $N \in R^{p\times p}$ both are invertible matrices, $U \in OR^{(n-k)\times(n-k)}$, $V \in OR^{k\times k}$, $P \in OR^{(n-k)\times(n-k)}$, $Q \in OR^{k\times k}$ are all orthogonal square matrices, and

$$\Sigma_{A_1} = \begin{pmatrix} I_{A_1} & 0 & 0 \\ 0 & S_{A_1} & 0 \\ 0 & 0 & 0_{A_1} \\ 0 & 0 & 0 \end{pmatrix}\begin{matrix} r \\ s \\ k-r-s \\ m-k \end{matrix},\quad \Sigma_{A_2} = \begin{pmatrix} 0_{A_2} & 0 & 0 \\ 0 & S_{A_2} & 0 \\ 0 & 0 & I_{A_2} \\ 0 & 0 & 0 \end{pmatrix}\begin{matrix} r \\ s \\ k-r-s \\ m-k \end{matrix},\quad \Sigma_{B_1} = \underset{r'\quad s'\quad k'-r'-s'\quad p-k'}{\begin{pmatrix} I_{B_1} & 0 & 0 & 0 \\ 0 & S_{B_1} & 0 & 0 \\ 0 & 0 & 0_{B_1} & 0 \end{pmatrix}},\quad \Sigma_{B_2} = \underset{r'\quad s'\quad k'-r'-s'\quad p-k'}{\begin{pmatrix} 0_{B_2} & 0 & 0 & 0 \\ 0 & S_{B_2} & 0 & 0 \\ 0 & 0 & I_{B_2} & 0 \end{pmatrix}}$$

where I_{A_1}, I_{A_2}, I_{B_1}, I_{B_2} are the identity matrices of appropriate order, o_{A_1}, o_{A_2}, o_{B_1}, o_{B_2} are the zero matrices of appropriate order (may not be square matrix). where

$$\mathrm{S}_{\mathrm{A}_1} = diag(\alpha_1, \alpha_2, \cdots, \alpha_s),\quad 1 > \alpha_1' \ge \alpha_2' \ge \cdots \ge \alpha_s' > 0;\ \mathrm{S}_{\mathrm{A}_2} = diag(\beta_1, \beta_2, \cdots, \beta_s),\quad 0 < \beta_1 \le \beta_2 \le \cdots \le \beta_s < 1;$$

$$\mathrm{S}_{\mathrm{B}_1} = diag(\alpha_1', \alpha_2', \cdots, \alpha_{s'}'),\quad 1 > \alpha_1' \ge \alpha_2' \ge \cdots \ge \alpha_{s'}' > 0;\ \mathrm{S}_{B_2} = diag(\beta_1', \beta_2', \cdots, \beta_{s'}'),\quad 0 < \beta_1' \le \beta_2' \le \cdots \le \beta_{s'}' < 1;$$

$$\mathrm{S}_{\mathrm{A}_1}^2 + S_{A_2}^2 = I_{\mathrm{S}},\ \mathrm{S}_{B_1}^2 + S_{B_2}^2 = I_{\mathrm{S}'},\quad k = rank\begin{bmatrix} A_1, & A_2 \end{bmatrix},\quad r = k - rank(A_2),\quad s = rank(A_1) + rank(A_2) - k,$$

$$k' = rank\begin{bmatrix} B_1, & B_2 \end{bmatrix},\quad r' = k' - rank(B_2^T),\quad s' = rank(B_1) + rank(B_2) - k'.$$

Canonical Correlation Decomposition (CCD[3][4] **)** Let $A_1 \in R^{m\times(n-k)}$, $A_2 \in R^{m\times k}$, $B_1 \in R^{p\times(n-k)}$, $B_2 \in R^{p\times k}$,and $rank(A_1) \ge rank(A_2)$,

$rank(B_1) \ge rank(B_2)$, Then, the CCD of matrix pair (A_1, A_2), (B_1, B_2) respectively are

$$A_1 = P_1(\overline{\Sigma}_{A_1}, 0)E_{A_1}^{-1},\quad A_2 = P_1(\overline{\Sigma}_{A_2}, 0)E_{A_2}^{-1};\quad B_1 = Q_1(\overline{\Sigma}_{B_1}, 0)E_{B_1}^{-1},\quad B_2 = Q_1(\overline{\Sigma}_{B_2}, 0)E_{B_2}^{-1} \tag{2}$$

where $P_1 \in OR^{m\times m}$, $Q_1 \in OR^{p\times p}$, $E_{A_1} \in R^{(n-k)\times(n-k)}$, $E_{A_2} \in R^{k\times k}$, $E_{B_1} \in R^{(n-k)\times(n-k)}$, $E_{B_2} \in R^{k\times k}$,

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$$\overline{\sum}_{A_1}=\begin{pmatrix} I_{r_1} & 0 & 0 \\ 0 & C_{A_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & D_{A_1} & 0 \\ 0 & 0 & I_{f_1} \end{pmatrix}, \quad \overline{\sum}_{A_2}=\begin{pmatrix} I_{r_1} & 0 & 0 \\ 0 & I_{s_1} & 0 \\ 0 & 0 & I_{h_1-r_1-s_1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \overline{\sum}_{B_1}=\begin{pmatrix} I_{r_2} & 0 & 0 \\ 0 & C_{B_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & D_{B_1} & 0 \\ 0 & 0 & I_{f_2} \end{pmatrix} \quad \overline{\sum}_{B_2}=\begin{pmatrix} I_{r_2} & 0 & 0 \\ 0 & I_{s_2} & 0 \\ 0 & 0 & I_{h_2-r_2-s_2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$C_{A_1}=diag(\lambda_1,\lambda_2,\cdots,\lambda_{s_1})>0$, $1>\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{s_1}>0$

$D_{A_1}=diag(\mu_1,\mu_2,\cdots,\mu_{s_1})>0$, $0<\mu_1\leq\mu_2\leq\cdots\leq\mu_{s_1}<1$, $\lambda_i^2+\mu_i^2=1$, $i=1,2,\cdots,s_1$;

$C_{B_1}=diag(\lambda_1^{'},\lambda_2^{'},\cdots,\lambda_{s_2}^{'})>0$, $1>\lambda_1^{'}\geq\lambda_2^{'}\geq\cdots\geq\lambda_{s_2}^{'}>0$

$D_{B_1}=diag(\mu_1^{'},\mu_2^{'},\cdots,\mu_{S_2}^{'})>0$, $0<\mu_1^{'}\leq\mu_2^{'}\leq\cdots\leq\mu_{s_2}^{'}<1$, $\lambda_j^{'2}+\mu_j^{'2}=1$, $j=1,2,\cdots,s_2$;

$g_1=rank(A_1)=r_1+s_1+f_1$, $s_1=rank(A_1,A_2)+rank(A_2A_1^T)-rank(A_1)-rank(A_2),$

$h_1=rank(A_2)$, $r_1=rank(A_1)+rank(A_2)-rank(A_1,A_2)$,

$g_2=rank(B_1)=r_2+s_2+f_2$, $s_2=rank(B_1,B_2)+rank(B_2^TB_1)-rank(B_1)-rank(B_2),$

$h_2=rank(B_2)$, $r_2=rank(B_1)+rank(B_2)-rank(B_1,B_2)$.

Problem I Given $A\in R^{m\times p}$, $B\in R^{n\times p}$, $C\in R^{p\times q}$, $D\in R^{p\times l}$, $E\in R^{m\times q}$, $F\in R^{n\times l}$, $G\in R^{m\times l}$, $H\in R^{n\times q}$,let

$S_E=\{X\,|\,X\in R^{p\times p},\|(AXC,BXD,AXD,BXC)-(E,F,G,H)\|=\min\}$,

find $\hat{X}\in S_E$, such that $\|\hat{X}\|=\min_{X\in S_E}\|X\|$.

2. The Solution of Optimal Approximation Problem

Lemma 1[1] Let $A\in R^{m\times p}$, $B\in R^{n\times p}$, $C\in R^{p\times q}$, $D\in R^{p\times l}$, $E\in R^{m\times q}$, $F\in R^{n\times l}$, $G\in R^{m\times l}$, $H\in R^{n\times q}$, the GSVD of matrix pair $\begin{pmatrix} A \\ B \end{pmatrix}$, $(C,\ D)$ are the corresponding form of equation (1). Let

$$U^TEP=(E_{ij})_{3\times 3},\ V^TFQ=(F_{ij})_{3\times 3},\ U^TGQ=(G_{ij})_{3\times 3},\ V^THP=(H_{ij})_{3\times 3}, \tag{3}$$

then, the least squares solution of $\|(AXC,BXD,AXD,\ BXC)-(E,F,G,H)\|=\min$ is

$$X=N^{-1}\begin{pmatrix} E_{11} & E_{12}S_C+G_{12}S_D & G_{13} & X_{14} \\ S_A^{-1}E_{21} & [X_{22}] & S_BF_{23}+S_AG_{23} & X_{24} \\ X_{31} & F_{32}S_D^{-1}+H_{32}S_C & F_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix}M^{-1}, \tag{4}$$

where $[X_{22}]=R*(S_AE_{22}S_C+S_BF_{22}S_D+S_AG_{22}S_D+S_EH_{22}S_C)$, $R=(\gamma_{ij})$, $\gamma_{ij}=\dfrac{1}{\alpha_i^2\alpha_j^{'2}+\beta_i^2\beta_j^{'2}+\alpha_i^2\beta_j^{'2}+\alpha_j^{'2}\beta_i^2}=1$ $1\leq i\leq s, 1\leq j\leq s^{'}$.

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Lemma 2[5] **(Projection Theorem)** Let X is the finite dimensional inner product space, M is the subspace of the inner product X, for a given $x \in X$, if exists $m_0 \in M$,such that $\|x-m_0\| \le \min\|x-m\|$, $\forall m \in M$,then m_0 is unique, and m_0 is the unique minimum vector of M if anf only if $x - m_0 \perp M$.

Lemma 3 Let $A \in R^{m\times p}$, $B \in R^{n\times p}$, $C \in R^{p\times q}$, $D \in R^{p\times l}$, $E \in R^{m\times q}$, $F \in R^{n\times l}$, $G \in R^{m\times l}$, $H \in R^{n\times q}$, X_0 is The least squares solution of matrix equations

$$(AXC, BXD, AXD, BXC) = (E, F, G, H) \ , \tag{5}$$

let

$$AX_0C = E_0 \ , BX_0D = F_0 \ , AX_0D = G_0 \ , BX_0C = H_0 \tag{6}$$

then the least squares solution set of compatible matrix equations

$$(AXC, BXD, AXD, BXC) = (E_0, F_0, G_0, H_0) \ , \tag{7}$$

Is the same as matrix equations (5).

Lemma 4[6] Let $J_1, J_2, J_3, J_4 \in R^{m\times n}$, $S_a = diag(a_1, a_2, \cdots, a_m) > 0$; $S_b = diag(b_1, b_2, \cdots, b_n) > 0$;

$S_c = diag(c_1, c_2, \cdots, c_m) \ge 0$; $S_e = diag(e_1, e_2, \cdots, e_m) \ge 0$; $S_f = diag(f_1, f_2, \cdots, f_n) \ge 0$;

$S_g = diag(g_1, g_2, \cdots, g_m) \ge 0$; $S_h = diag(h_1, h_2, \cdots, h_n) \ge 0$.

So there is a unique $S \in R^{m\times n}$ that makes

$$\Phi = \|S_aSS_b - J_1\|^2 + \|S_cSS_d - J_2\|^2 + \|S_eSS_f - J_3\|^2 + \|S_gSS_h - J_4\|^2 = \min \ .$$

And it has an expression

$$S = P * (S_aJ_1S_b + S_cJ_2S_d + S_eJ_3S_f + S_gJ_4S_h) \ , P = (p_{ij}) \in R^{m\times m} \ , \quad p_{ij} = \frac{1}{a_i^2b_j^2 + c_i^2d_j^2 + e_i^2f_j^2 + g_i^2h_j^2} \ , 1 \le i \le m \ , 1 \le j \le n \ .$$

Note. Lemma 4 is a corollary of Lemma 2.1 in reference [2], in which $S_i (i = a,b,c,d,e,f,h)$ Can be zero diagonal matrix or identity matrix.

Theorem 1 Let $A \in R^{m\times p}$, $B \in R^{n\times p}$, $C \in R^{p\times q}$, $D \in R^{p\times l}$, $E \in R^{m\times q}$, $F \in R^{n\times l}$, $G \in R^{m\times l}$, $H \in R^{n\times q}$, and the CCD of matrix pair (A^T, B^T), (C,D) are the corresponding form of equation (2). Let

$$E_A^TEE_C = (\overline{E}_{ij})_{4\times4} \ , E_B^TFE_D = (\overline{E}_{ij})_{4\times4} \ , E_A^TGE_D = (\overline{G}_{ij})_{4\times4} \ , E_B^THE_C = (\overline{H}_{ij})_{4\times4} \ , P^TXQ = (X_{ij})_{6\times6} \tag{8}$$

$$E_A = (\underset{r_1}{A_1} \ \ \underset{s_1}{A_2} \ \ \underset{f_1}{A_3} \ \ \underset{m-g_1}{A_4}) \ , \quad E_B = (\underset{r_2}{B_1} \ \ \underset{s_2}{B_2} \ \ \underset{h_1-r_1-s_1}{B_3} \ \ \underset{n-h_1}{B_4}) \ E_C = (\underset{r_2}{C_1} \ \ \underset{s_2}{C_2} \ \ \underset{f_2}{C_3} \ \ \underset{l-g_2}{C_4}) \ , \quad E_D = (\underset{r_2}{D_1} \ \ \underset{s_2}{D_2} \ \ \underset{h_2-r_2-s_2}{D_3} \ \ \underset{l-h_2}{D_4})$$

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here $\overline{E}_{ij}=A_i^TEC_j$, $\overline{F}_{ij}=B_i^TFD_j$, $\overline{G}_{ij}=A_i^TGD_j$, $\overline{H}_{ij}=B_i^THC_j$, $i,j=1,2,3,4$,then the sufficient and necessary conditions for matrix equations (5) to have a compatible solution set are

$$\overline{E}_{i4}=0,\ \overline{E}_{4i}=0,\ \overline{F}_{i4}=0,\ \overline{F}_{4i}=0,\ \overline{G}_{i4}=0,\ \overline{G}_{4i}=0,\ \overline{H}_{i4}=0,\ \overline{H}_{4i}=0,\ i=1,2,3,4;$$

$$\overline{E}_{11}=\overline{F}_{11}=\overline{G}_{11}=\overline{H}_{11},\ \overline{F}_{12}=\overline{G}_{12},\ \overline{F}_{13}=\overline{G}_{13},\ \overline{E}_{21}=\overline{G}_{21},\ \overline{F}_{21}=\overline{H}_{21},\ \overline{E}_{31}=\overline{G}_{31},\ \overline{F}_{31}=\overline{H}_{31},,\ \overline{E}_{13}=\overline{H}_{13}. \tag{9}$$

And the general solution of X is

$$X=P\begin{pmatrix}\overline{E}_{11}&\overline{F}_{12}&\overline{F}_{13}&X_{14}&\widetilde{X}_{15}&\overline{E}_{13}\\ \overline{F}_{21}&\overline{F}_{22}&\overline{F}_{23}&X_{24}&\widetilde{X}_{25}&H_{23}\\ \overline{F}_{31}&\overline{F}_{32}&\overline{F}_{33}&X_{34}&\widetilde{X}_{35}&H_{33}\\ X_{41}&X_{42}&X_{43}&X_{44}&X_{45}&X_{46}\\ \widetilde{X}_{51}&\widetilde{X}_{52}&\widetilde{X}_{53}&X_{54}&\widetilde{X}_{55}&\widetilde{X}_{56}\\ \overline{E}_{31}&\overline{G}_{32}&\overline{G}_{33}&X_{64}&\widetilde{X}_{65}&\overline{E}_{33}\end{pmatrix}Q^T, \tag{10}$$

where $\widetilde{X}_{15}=(\overline{E}_{12}-\overline{F}_{12}C_C)D_C^{-1}$, $\widetilde{X}_{25}=(\overline{H}_{22}-\overline{F}_{22}C_C)D_C^{-1}$, $\widetilde{X}_{35}=(\overline{H}_{32}-\overline{F}_{32}C_C)D_C^{-1}$, $\widetilde{X}_{51}=D_A^{-1}(\overline{E}_{21}-C_A\overline{F}_{21})$, $\widetilde{X}_{52}=D_A^{-1}(\overline{G}_{22}-C_A\overline{F}_{22})$, $\widetilde{X}_{53}=D_A^{-1}(\overline{G}_{23}-C_A\overline{F}_{23})$, $\widetilde{X}_{55}=D_A^{-1}(\overline{E}_{22}-\overline{G}_{22}C_C-C_AH_{22}+C_AF_{22}C_C)D_C^{-1}$, $\widetilde{X}_{56}=D_A^{-1}(\overline{E}_{23}-C_AH_{23})$, $\widetilde{X}_{65}=(\overline{E}_{32}-\overline{G}_{32}C_C)D_C^{-1}$ 且 X_{i4} , X_{4i} $(i=1,2,\cdots,6)$ Is any matrix with appropriate size.

Theorem 2 Let $A\in R^{m\times p}$, $B\in R^{n\times p}$, $C\in R^{p\times q}$, $D\in R^{p\times l}$, $E\in R^{m\times q}$, $F\in R^{n\times l}$, $G\in R^{m\times l}$, $H\in R^{n\times q}$, the GSVD of matrix pair $\begin{pmatrix}A\\B\end{pmatrix}$, $(C,\ D)$ are the corresponding form of equation (1),block matrices E_{ij} , F_{ij} , G_{ij} , H_{ij} $(i,j=1,2,3,4)$ are given by equation (8), the free block matrices X_{i4} , X_{4i} $(i=1,2,3,4)$ in Equation (10) are all zero block matrices, denoted X as X_0 , Let X_0 be a least squares solution of the matrix equations (5), let $E_0=AX_0C$, $F_0=BX_0D$, $G_0=AX_0D$, $H_0=BX_0C$,then E_0 , F_0 , G_0 , H_0 can be expressed as

$$E_0=U\begin{pmatrix}E_{11}&E_{12}S_C^2+G_{12}S_DS_C&0\\ S_A^2E_{21}+S_AS_BH_{21}&S_A[X_{22}]S_C&0\\ 0&0&0\end{pmatrix}P^T,\quad F_0=V^T\begin{pmatrix}0&0&0\\ 0&S_B[X_{22}]S_D&S_B^2F_{23}+S_BS_AG_{23}\\ 0&F_{32}S_D^2+H_{32}S_CS_D&F_{33}\end{pmatrix}Q,$$

$$G_0=U\begin{pmatrix}0&E_{12}S_CS_D+G_{12}S_D^2&G_{13}\\ 0&S_A[X_{22}]S_D&S_AS_BF_{23}+S_A^2G_{23}\\ 0&0&0\end{pmatrix}Q^T,\quad H_0=V\begin{pmatrix}0&0&0\\ S_BS_AE_{21}+S_B^2H_{21}&S_B[X_{22}]S_C&0\\ H_{31}&F_{32}S_DS_C+H_{32}S_C^2&0\end{pmatrix}P^T.$$

Theorem 3 Let $A\in R^{m\times p}$, $B\in R^{n\times p}$, $C\in R^{p\times q}$, $D\in R^{p\times l}$, $E\in R^{m\times q}$, $F\in R^{n\times l}$, $G\in R^{m\times l}$, $H\in R^{n\times q}$, E_0 , F_0 , G_0 , H_0 are given by theorem 2, let

$$E_A^TE_0E_C=(\widetilde{E}_{ij})_{4\times4},\ E_B^TF_0E_D=(\widetilde{F}_{ij})_{4\times4},\ E_A^TG_0E_D=(\widetilde{G}_{ij})_{4\times4}, \tag{11}$$

Then, the matrix equations (5) have a unique minimal norm least square solution $\hat{X}$ as

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$$\hat{X} = P\begin{pmatrix} \widetilde{E}_{11} & \widetilde{F}_{12} & \widetilde{F}_{13} & 0 & \widehat{X}_{15} & \widetilde{E}_{13} \\ \widetilde{F}_{21} & \widetilde{F}_{22} & \widetilde{F}_{23} & 0 & \widehat{X}_{25} & \widetilde{H}_{23} \\ \widetilde{F}_{31} & \widetilde{F}_{32} & \widetilde{F}_{33} & 0 & \widehat{X}_{35} & \widetilde{H}_{33} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \widehat{X}_{51} & \widehat{X}_{52} & \widehat{X}_{53} & 0 & \widehat{X}_{55} & \widehat{X}_{56} \\ \widetilde{E}_{31} & \widetilde{G}_{32} & \widetilde{G}_{33} & 0 & \widehat{X}_{65} & \widetilde{E}_{34} \end{pmatrix} Q^T, \tag{12}$$

where $\widehat{X}_{15} = D_A^{-1}(\widetilde{H}_{22} - \widetilde{F}_{22}C_C)D_C^{-1}$, $\widehat{X_{25}} = (\widetilde{H}_{21} - \widetilde{F}_{22}C_C)D_C^{-1}$, $\widehat{X}_{35} = (\widetilde{H}_{32} - \widetilde{F}_{32}C_C)D_C^{-1}$, $\widehat{X}_{51} = D_A^{-1}(\widetilde{E}_{21} - C_A\widetilde{F}_{21})$

$\widehat{X}_{52} = D_A^{-1}(\widetilde{G}_{22} - C_A\widetilde{F}_{22})$, $\widehat{X}_{53} = D_A^{-1}(\widetilde{G}_{23} - C_A\widetilde{F}_{23})$, $\widehat{X}_{55} = D_A^{-1}(\widetilde{E}_{22} - \widetilde{G}_{22}C_C - C_A\widetilde{F}_{22}C_C)D_C^{-1}$, $\widehat{X}_{56} = D_A^{-1}(\widetilde{E}_{23} - C_A\widetilde{H}_{23})$,

$\widehat{X}_{65} = (\widetilde{E}_{32} - \widetilde{G}_{32}C_C)D_C^{-1}$.

Proof. By theorem 1 ,let X_1 as any solution of the compatible matrix equations(7), then X_1 can be expressed as (10) in the form of Theorem 1, so $X_{i4} = 0$, $X_{4i} = 0$ (i=1,2,⋯,6) by $\|X_1\|^2 = \min$, substitute the results into the formula X_1 respectively, and let X_1 be $\widehat{X}$, by Lemma 2, $\widehat{X}$ is the minimal norm least squares solution of the matrix equations $(AXC, BXD, AXD, BXC) = (E, F, G, H)$,that is, the solution $\widehat{X}$ of problem I is Equation (12).

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