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On complex analysis and Riemann zeta-function

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Abstract

Counterexamples are given to residue theorem of complex analysis. The functional equation of Riemann zeta-function is invalid.

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1. Introduction

Complex Analysis(CA) dated back to the time of Gauss, after he put a square-root of -1 on the 2-dimensional Euclidean space, whose action would be unreasonable, although later on CA has been developed by Riemann, Cauchy, et al, making CA to be a rich branch of mathematics. Recently in [L1] a manner of constructing a counterexample to CA was introduced, however [L1] used the invalid equality(see (7) of p.62 of [D], and (1) of p.189 of [L1])

(1) $$s(s-1)\pi^{-0.5s}\Gamma(0.5s)\zeta(s)$$

$$=1+s(s-1)\left(\int_1^{\infty}\left(x^{0.5s-1}+x^{-0.5s-0.5}\right)\omega(x)dx\right),$$

$$\omega(x)=\sum_1^{\infty}e^{-\pi n^2 x},$$

because to deduce (1) one must use the below invalid equation (2) of the present paper(see pp.61-62 of [D]). Here we need to speak more about (1). (1) is certainly incorrect, for in conjunction with the work of [L1](see the arguments given on pp.192-193, especially (2) of [L1]), the validity of (1) can result in the absurd equality "$1=0$". Precisely speaking, in (2) of [L1] now we know that $\rho_2=8\pi(7.5\beta+4\pi^2\delta-15\pi\gamma)\neq 0$(see (7) below of the present paper; symbols are the same as in [L1]), and thus we need not to consider the coefficient of t^{-3} of (2) of [L1](we can estimate simply the term containing t^{-3} as $O(t^{-3})$; here we note that the coefficient of t^{-3} given by (2) of [L1] is incorrect), and the arguments of [L1] then indicate that if (1) is valid then we can obtain the estimate $1+O(t^{-1})=O\left(t^3e^{-t/5}\right)$ for any large t(see the situation "$\rho_1=0,\rho_2\neq 0$" given on p.192 of [L1]), which yields the absurd equality "$1=0$" after letting $t\to+\infty$. The "functional equation" of ζ function is, for $\xi(z)=0.5z(z-1)\pi^{-0.5z}\Gamma(0.5z)\zeta(z)$,

$$\xi(s)=\xi(1-s).$$

(see p.62, [D]). This follows only from using (1). Due to the invalidity of (1), the "functional equation" is now invalid also.

In this paper, we shall give a simple method to construct a convincing counterexample, showing(see below) the invalidity of the "residue theorem" of CA, which makes results of number theory obtained by using CA to be invalid. Our work needs the help of a suitable calculator(our calculator has the accuracy up to 10^{-15}).

2. The construction of the counterexample

By (6) of p.62 of Davenport's book [D], we have

(2) $$x^{0.5}\left(1+2\sum_1^{\infty}e^{-\pi n^2 x}\right)=1+2\left(\sum_1^{\infty}e^{-\pi n^2/x}\right),$$

for all $x>0$. Note that this formula is derived by means of the "residue theorem" of complex analysis, where one needs to move lines of integrations on the complex plane, see pp.63-64. Our purpose is to deduce a contradiction, by assuming the validity of (2), which in turn shows the invalidity of (2), and also shows the invalidity of the "residue theorem", because the Possion's summation formula part of p.63 of [D] can be replaced by the usual Fourier series treatment, as disclosed in the proof of Lemma 1.3.3(it is just (8) of p.63 of [D]) of [L2].

Assuming (2). Taking derivatives on both sides of the above equality, we get

$$0.5x^{-0.5}\left(1+2\sum_1^\infty e^{-\pi n^2 x}\right)+x^{0.5}(-2\pi)\left(\sum_1^\infty n^2 e^{-\pi n^2 x}\right)$$

$$=(2\pi x^{-2})\left(\sum_1^\infty n^2 e^{-\pi n^2/x}\right),$$

and thus

$$0.5x^{-0.5}\left(1+2\sum_1^\infty e^{-\pi n^2 x}\right)=(2\pi x^{-2})\left(\sum_1^\infty n^2 e^{-\pi n^2/x}\right)$$

(3) $$+\left(2\pi x^{0.5}\right)\left(\sum_1^\infty n^2 e^{-\pi n^2 x}\right).$$

Taking derivatives on both sides of (3), we get

$$-0.25x^{-1.5}\left(1+2\sum_1^\infty e^{-\pi n^2 x}\right)-2\pi x^{-0.5}\left(\sum_1^\infty n^2 e^{-\pi n^2 x}\right)$$

$$+2\pi^2 x^{0.5}\left(\sum_1^\infty n^4 e^{-\pi n^2 x}\right)=-4\pi x^{-3}\left(\sum_1^\infty n^2 e^{-\pi n^2/x}\right)+$$

(4) $$+2\pi^2 x^{-4}\left(\sum_1^\infty n^4 e^{-\pi n^2/x}\right).$$

We use (3) to substitute the first term on the left side of (4), getting

$$-0.5x^{-1}\left((2\pi x^{-2})\left(\sum_1^\infty n^2 e^{-\pi n^2/x}\right)+\left(2\pi x^{0.5}\right)\left(\sum_1^\infty n^2 e^{-\pi n^2 x}\right)\right)$$

$$-2\pi x^{-0.5}\left(\sum_1^\infty n^2 e^{-\pi n^2 x}\right)+2\pi^2 x^{0.5}\left(\sum_1^\infty n^4 e^{-\pi n^2 x}\right)$$

$$=-4\pi x^{-3}\left(\sum_1^\infty n^2 e^{-\pi n^2/x}\right)+2\pi^2 x^{-4}\left(\sum_1^\infty n^4 e^{-\pi n^2/x}\right),$$

and thus

(5) $$3F(x)=2\pi G(x),$$

where

$$F(x)=-x^{-0.5}\left(\sum_1^\infty n^2 e^{-\pi n^2 x}\right)+x^{-3}\left(\sum_1^\infty n^2 e^{-\pi n^2/x}\right),$$

$$G(x)=\sum_1^\infty x^{-4} n^4 e^{-\pi n^2/x}-x^{0.5}\left(\sum_1^\infty n^4 e^{-\pi n^2 x}\right).$$

We have

$$F'(x) = 0.5x^{-1.5}(\sum_1^\infty n^2 e^{-\pi n^2 x}) + \pi x^{-0.5}(\sum_1^\infty n^4 e^{-\pi n^2 x})$$

$$-3x^{-4}(\sum_1^\infty n^2 e^{-\pi n^2/x}) + \pi x^{-5}(\sum_1^\infty n^4 e^{-\pi n^2/x}),$$

$$G'(x) =- 4x^{-5}(\sum_1^\infty n^4 e^{-\pi n^2/x}) + \pi x^{-6}(\sum_1^\infty n^6 e^{-\pi n^2/x})$$

$$- 0.5x^{-0.5}(\sum_1^\infty n^4 e^{-\pi n^2 x}) + \pi x^{0.5}(\sum_1^\infty n^6 e^{-\pi n^2 x}),$$

and from (5) we get

(6) $$3F'(x) = 2\pi G'(x).$$

We choose $x = 1$ in (6) to get

$$3(0.5\beta + \pi\gamma - 3\beta + \pi\gamma) = 2\pi(-4.5\gamma + 2\pi\delta),$$

where

$\beta = \sum_1^\infty n^2 e^{-\pi n^2}, \gamma = \sum_1^\infty n^4 e^{-\pi n^2}, \delta = \sum_1^\infty n^6 e^{-\pi n^2},$

which gives

$$15\pi\gamma = 7.5\beta + 4\pi^2\delta.$$

But we shall prove that actually

(7) $$15\pi\gamma > 7.5\beta + 4\pi^2\delta.$$

Thus we conclude that (2) must be invalid.

For verifying (7), we note that

$\pi = 3.141592653589793..., e = 2.718281828459045...,$

and thus

(8) $$15\pi\gamma - (7.5\beta + 4\pi^2\delta) > a + b + c + d,$$

$$a = e^{-\pi}(15\pi - 7.5 - 4\pi^2) > 0.006286423781,$$

$$b = e^{-4\pi}(240\pi - 30 - 256\pi^2) >- 0.006286410606,$$

$$a + b > 10^{-8} \times 1.317,$$

$$c = (1215\pi - 67.5 - 2916\pi^2)e^{-9\pi} >- 1.316 \times 10^{-8},$$

(9) $$a + b + c > 10^{-11},$$

and

(10) $$d =-\sum_{n\geq 4}(4\pi^2 n^6)e^{-\pi n^2}.$$

To estimate d, we need to prove

(11) $$4\pi^2X^6 < (0.000001)e^{\pi X^2-\pi X}, \text{ for } X \geq 4.$$

To verify (11), we first note that (11) is true for $X = 4-\varepsilon$, by taking the logarithm(ε is a sufficiently small positive number). Thus (11) can be verified, if one can show that(using the method of verifying the monotonic property)

(12) $$(10^{-6})e^{\pi X^2-\pi X} > 24\pi X^4, \text{ for any } X > 4-2\varepsilon.$$

Similarly, to verify (12) it suffices to verify

(13) $$(10^{-6})e^{\pi X^2-\pi X} > 96X^2, \text{ for any } X > 4-3\varepsilon.$$

And, to verify (13) it suffices to verify

(14) $$(10^{-6})\pi e^{\pi X^2-\pi X} > 192, \text{ for any } X > 4-4\varepsilon.$$

Obviously (14) holds. Thus (11) is proved. By (10) and (14) we have

$$d >-\sum_{n\geq 4}(10^{-6})e^{-\pi n} > -(10^{-6})\left(\sum_{n\geq 4}e^{-\pi n}\right)$$

$$=- 10^{-6}\left(\frac{e^{-4\pi}}{1-e^{-\pi}}\right) >- 3.65\times 10^{-12},$$

and the inequality (7) follows from (8) and (9).

Remark 1. Using the similar method, it may be possible to show that $\rho_1 = 1+2\alpha-8\pi\beta \neq 0$, here(using the symbols of p.192 of [L1])

$$\alpha = \sum_1^{\infty} e^{-\pi n^2}, \quad \beta = \sum_1^{\infty} n^2 e^{-\pi n^2},$$

and if this can be verified, then we can derive our assertions(the invalidity of (2), (1), and the "residue theorem") more easily, see p.192 of [L1]; note that by letting $x = 1$ in (3), we get "$1+2\alpha-8\pi\beta = 0$".

3. Another counterexample to the "residue theorem" of CA

Here we shall give another example, for the purpose of negating the "residue theorem" of complex analysis. Let

$$I = \int_0^{\infty}\frac{sinx}{x^{0.5}}dx.$$

On the one hand, by using the "residue theorem" we can deduce that

(15) $$I = \sqrt{\frac{\pi}{2}} = 1.253314137315500\ldots.$$

Whereas on the other hand a direct calculation using a calculator reveals that actually

(16) $$\int_0^{\infty} \frac{sinx}{x^{0.5}} dx > \sqrt{\frac{\pi}{2}},$$

and the difference of the two values is greater than 7×10^{-13}. A comparison of (15) and (16) then yields our another counterexample to the "residue theorem" of CA.

First we derive (15). The details are taken from the proof of Lemma 1.4.1 of [L3], which we present at here for the sake of completeness. Let L be the closed path of four-sides trapezoid on the complex plane, consisting of the four vertexes $(N, 0)$, $(M, 0)$, (M, M), and (N, N), here $M > N > 0$. By the "residue theorem" we have (here $e(t) = exp(2\pi i t) = e^{2\pi i t}$)

(17) $$\int_N^M e(x^2) dx = R_1 + R_2 + R_3,$$

where R_1, R_2 and R_3 are the complex linear integrals of the function $e(z^2)$ along the straight lines from (M, M) to $(M, 0)$, from (N, N) to (M, M), and from $(N, 0)$ to (N, N), respectively. Let M be sufficiently large, and N be sufficiently small, then

$$R_1 = O\left(\int_0^M exp\,(-4\pi M y)\, dy\right) = O\left(\frac{1}{M}\right),$$

$$R_2 = (1+i)\left(\int_N^M exp(2\pi i(t+ti)^2) dt\right) = (1+i)\left(\int_N^M exp(-4\pi t^2) dt\right),$$

$$R_3 = O\left(\int_0^N exp\,(-4\pi N y)\, dy\right) = O(N),$$

and thus, letting $M \to \infty$ and $N \to 0$ in (17) we get

(18) $$\int_0^{\infty} e(x^2) dx = (1+i)W, W = \int_0^{\infty} exp(-4\pi t^2) dt.$$

The value $W = 0.25$ can be obtained by a familiar method. Let ε be a sufficiently small positive number. Then

(19) $$W^2 = \int_0^{\infty}\int_0^{\infty} = \iint_D exp\left(-4\pi(x^2 + y^2)\right) dxdy + O(\varepsilon),$$

where $D = \{(x, y): 0 \le x < \infty, 0 \le y < \infty, x^2 + y^2 \ge \varepsilon^2\}$. Using the substitution of variables $x = rcos\theta, y = rsin\theta, r \ge \varepsilon, 0 \le \theta \le 0.5\pi$, we have

$D = \{(rcos\theta, rsin\theta): \varepsilon \leq r < \infty, 0 \leq \theta \leq 0.5\pi\}$,

and thus

$W^2 = \int_{\varepsilon}^{\infty} \int_0^{0.5\pi} exp(-4\pi r^2) r dr d\theta = \frac{1}{16} exp(-4\pi\varepsilon^2)$.

Letting $\varepsilon \to 0$, from (18) and (19) we get $W = 0.25$. Then, using a substitution of the variables, we obtain

$\int_0^{\infty} e^{ix} x^{-0.5} dx = (\sqrt{2\pi}) \int_0^{\infty} e(y) y^{-0.5} dy$

$= 2(\sqrt{2\pi}) \int_0^{\infty} e(x^2) dx = 2(\sqrt{2\pi})(1 + i)W$,

which implies (15).

Next we verify (16). Let $A = 1200.0883936$, and

$$B = 382\pi - A = 0.000000071300\ldots = 10^{-8} \times 7.1300\ldots.$$

We find that

(20) $$\int_0^A \frac{sinx}{x^{0.5}} dx = 1.224447702040165\ldots.$$

Note that $cosA = cosB, sinA = - sinB$. Integrating by parts several times, we get

(21) $$\int_A^{\infty} \frac{sinx}{x^{0.5}} dx = (cosB) \cdot A^{-0.5}$$

$$-0.5(sinB)A^{-1.5} - 0.75(cosB)A^{-2.5} + 1.875(sinB)A^{-3.5}$$

$$+ 6.5625(cosB)A^{-4.5} - \frac{945}{32}(sinB)A^{-5.5} + \frac{945}{32}\left(\int_A^{\infty} (-5.5)(sinx)x^{-6.5} dx\right) = \sum_{1 \leq i \leq 7} \psi_i, \text{ say.}$$

Using the Taylor expansion we know that

$1 - \frac{B^2}{2} < cosB < 1, B - \frac{B^3}{6} < sinB < B$,

and consequently (21) gives

$\psi_1 = P_1 + \Delta_1, P_1 = A^{-0.5}, |\Delta_1| < \frac{B^2}{2} \cdot A^{-0.5} < 10^{-15}$,

$\psi_2 = P_2 + \Delta_2, P_2 = - 0.5BA^{-1.5}, |\Delta_2| < \frac{B^3}{12} \cdot A^{-1.5} < 10^{-15}$,

$\psi_3 = P_3 + \Delta_3, P_3 = - 0.75A^{-2.5}, |\Delta_3| < 0.375B^2A^{-2.5} < 10^{-15}$,

$$0 < \psi_4 < 10^{-15},$$

$\psi_5 = P_5 + \Delta_5$, $P_5 = 6.5625A^{-4.5}$, $|\Delta_5| < 4B^2A^{-4.5} < 10^{-15}$,

$|\psi_6| \leq \frac{945}{32}BA^{-5.5} < 10^{-15}$,

$|\psi_7| \leq \frac{945}{32}A^{-5.5} < 10^{-15}$,

and hence

(22) $$\int_A^\infty \frac{sinx}{x^{0.5}}dx = P_1 + P_2 + P_3 + P_5 + \theta \cdot (7 \times 10^{-15}),$$

here $|\theta| < 1$. We have

$P_1 = 0.028866450308450 \ldots$,

$$P_2 =- 10^{-13} \times 8.575109 \ldots,$$

$P_3 =- 10^{-8} \times 1.503239484 \ldots$,

$P_5 = 10^{-13} \times 0.9132 \ldots$,

$$P_1 + P_2 + P_3 + P_5 = 0.028866435276112 \ldots,$$

which gives, by means of (20) and (22), the numerical value

$\int_0^\infty \frac{sinx}{x^{0.5}}dx = 1.253314137316277 \ldots + \theta \cdot (7 \times 10^{-15})$,

and therefore the proof of (16) is finished.

Remark 2. In view of our present research, it is natural to suspect many "identities" involving fractions which are guessed by Ramanujan, such as

$\frac{1}{\pi} = \frac{2\sqrt{2}}{99^2}\sum_{k=0}^\infty \frac{(4k)!}{(k!)^4} \cdot \frac{26390k+1103}{(396)^{4k}}$,

for using a powerful calculator one may find that the two sides are not the same, requiring the sufficient accuracy. We have to admit that sometimes a computer can win a chess championship, and mention the solution to the "four-colour problem". It is during such a time that it is possible to disclose the drawback of works of previous scholars.

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