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Nonlinear H_∞ Control of Delayed Recurrent Neural Networks Influenced by Uncertain Noise

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Abstract:

This paper presents a theoretical design of how a nonlinear H_∞ optimal control is achieved for delayed recurrent neural networks with noise uncertainty. Our objective is to build globally stabilizing control laws to accomplish the input-to-state stability together with the optimality for delayed recurrent neural networks, and to attenuate noises to a predefined level with stability margins. The formulation of H_∞ control is developed by using Lyapunov technique and solving a Hamilton-Jacobi-Isaacs (HJI) equation indirectly. To illustrate the analytical results, three numerical examples are given to demonstrate the effectiveness of the proposed approach.

Key Words: Delayed recurrent neural networks, nonlinear H_∞ optimal control, noise attenuation, input-to-state stability, Lyapunov technique, Hamilton-Jacobi-Isaacs (HJI) equation.

1. Introduction

The past two decades have witnessed enormous advances in building artificial computational systems toward real world applications, among which the system of recurrent neural networks has become one of the most promising systems used in many scientific areas, such as modeling biological sensor-motor systems, system identification and control, content- addressable memory, pattern recognition, image processing, and combinatorial optimization, etc. Many applications of them require a well-defined solution for all possible initial conditions under different circumstances. From a mathematical point of view, this signifies that the network should have a unique equilibrium point, which is both stable and globally attractive. Therefore, the study of both stability and controllability for recurrent neural networks has become an important issue in the last two decade, especially in the area of delayed recurrent neural networks, see, for example, [1-25], and reference therein.

Due to the presence of the disturbance in the real world, a real system is usually influenced by both internal and external random perturbations that are also called as noises. With respect to neural networks, the noise is an unavoidable factor that should be taken into consideration during the implementation of artificial neural networks. Therefore, in the past few years, the study of noisy recurrent neural networks (also known as stochastic recurrent neural networks) has started to attract the attention from the research community. Some interesting results can be found in [26-30]. However, most researchers in the literature deal with this problem by considering stochastic models and methodologies. Few authors have handled the problem from the perspective of deterministic models and systems, with which simpler approaches could be generated to attenuate the impact of the noise.

From the viewpoint of control theory, the noise (also called disturbance) attenuation belongs to the subject of robust nonlinear control problem, which has been widely studied as a nonlinear H_∞ control problem or a differential game problem, see, for example, [31-35], and reference therein. Because it is too difficult to solve the Hamilton-Jacobi-Isaacs (HJI) equation, the problem of finding a direct H_∞ optimal control solution for general nonlinear systems still remains open. Recently, the study of robust control for the stability of delayed recurrent neural network has been reported in [36] and [37]. The controllers developed in those two papers were constructed by using linear matrix inequality (LMI) approach. To the best of our knowledge, no publication presents results about nonlinear H_∞ optimal control for delayed continuous-time recurrent neural networks with noise uncertainty modeled by deterministic systems via solving a Hamilton-Jacobi-Isaacs equation.

Motivated by the discussions above and based on our previous study [30] for stochastic neural networks, this paper presents how a nonlinear H_∞ optimal control is achieved for deterministic noisy time-delay recurrent neural networks. The main contributions and the originality of our research are as follows: (1) A nonlinear H_∞ optimal control is developed for noisy time-delay recurrent neural networks modeled by a deterministic system, which is a simpler approach than the one developed in [30] to handle the impact of the noise. (2) A state feedback stabilizing control is also developed to guarantee that the system of noisy time-delay recurrent neural networks is input-to-state stability, which is a stronger control than the one developed in [30] with respect to stability margins. In addition, three examples are used to demonstrate the effectiveness of our approach. The rest of the paper is organized as follows. In Section 2, we present the problem formulation and mathematical preliminaries. In Section 3, we detail the theoretical results. In Section 4, we demonstrate the performance of our design with two numerical examples. Finally, the conclusion of the paper is given in Section 5.

2. Problem Formulation and Preliminaries

Based on the standard formulation given by [38], we consider the following deterministic model of noisy time-delay recurrent neural networks, which is derived from the most common model of delayed recurrent neural networks [39] in the literature plus an unknown noise signal Ψ_i

$$\dot{x}_i(t) = -\lambda x_i(t) + \sum_{j=1}^{n} w_{ij}^1 f_j(x_j(t)) + \sum_{j=1}^{n} w_{ij}^2 g_j(x_j(t-\tau)) + u_i + \Psi_i \tag{1}$$

where $i = 1,2,\cdots,n$. Mathematically, this can be described by the following matrix-vector compact form

$$\dot{x}(t) = -Ax(t) + W_1 f(x(t)) + W_2 g(x(t-\tau)) + u + \Psi \tag{2}$$

where $x(t) \in R^n$ is the state of the time-delay recurrent neural network, $u \in R^n$ is the input, $A = diag(\lambda, \cdots, \lambda) = \lambda I \in R^{n \times n}$ and $\lambda > \frac{1}{8}\gamma^{-2} > 0$ (γ is a design parameter), both $f(x(t)) =$ $\left[f_1(x_1(t)), \cdots, f_j(x_j(t)), \cdots, f_n(x_n(t))\right]^T \in R^n$ and $g(x(t-\tau)) = \left[g_1(x_1(t-\tau)), \cdots, g_j(x_j(t-\tau)), \cdots, g_n(x_n(t-\tau))\right]^T$ $\in R^n$ are vector functions, in addition, $f_j(x_j(t))$ and $g_j(x_j(t-\tau))$ are sigmoidal functions (scalar ones) that model the nonlinear input-output activations, $W_1 \in R^{n \times n}$ and $W_2 \in R^{n \times n}$ are weight matrices, $\tau \in R^+$ is the time delay, $\Psi \in R^n$ is an unknown noise signal that belongs to either L_2 or L_∞.

Remark 2.1: The model (2) represents a very general recurrent neural network model that includes the popular Hopfield neural networks, the paradigm of cellular neural networks, the bio-directional associative memory networks, and several other neural networks frequently employed in the literature.

Our design goal is to develop globally stabilizing control laws that include a nonlinear H_∞ optimal control for the delayed recurrent neural networks modeled by (2) to achieve the input-to-state stability together with the optimality, and to attenuate noises to a predefined level with stability margins.

Before we derive the main analytical results in the next section, it is necessary to introduce the following definitions that include a H_∞ control background.

Definition 2.1: The norm $\|x\|$ of a vector x is the Euclidean norm. If A is a matrix, then $\|A\|$ denotes the Frobenius matrix norm, defined as $\|A\| = (T_r\{A^T A\})^{1/2}$, where $T_r\{.\}$ denotes the trace of a matrix. L_2 and L_∞ denote the respective sets of measurable signals with finite norms $\|x(t)\|_2 = \left(\int_0^\infty x(t)^T x(t) dt\right)^{1/2}$ and $\|x(t)\|_\infty = \sup_{t \in [0,\infty)} \max_{1 \le i \le n} |x_i(t)|$. If $f : R^n \to R^n$ is a vector field and $V : R^n \to R$ is a scalar function, the notation $L_f V$ is used to denote $\frac{\partial V}{\partial x} f(x) = \frac{\partial V}{\partial x_1} f_1(x) \cdots + \frac{\partial V}{\partial x_i} f_i(x) \cdots + \frac{\partial V}{\partial x_n} f_n(x)$.

Definition 2.2: The activation functions $f_j(x_j(t))$ and $g_j(x_j(t))$ have the following properties:

(i) $f_j(0) = 0$ and $g_j(0) = 0$.

(ii) The scalar functions $f_j(x_j(t))$ and $g_j(x_j(t))$ are monotonically increasing and globally Lipschitz continuous with Lipschitz constants $h_j > 0$ and $k_j > 0$, i.e.

$$\left| f_j(x_i) - f_j(y_i) \right| \le h_j \left| x_i - y_i \right|$$

and

$$\left| g_j(x_i) - g_j(y_i) \right| \le k_j \left| x_i - y_i \right|$$

where $\forall x_i \in R,\ \forall y_i \in R$, and $|\ \cdot\ |$ represents the absolute value.

Definition 2.3 (H_∞ Control Theory**)**: (1) Linear System: As described in [34-35], the objective of the H_∞ design for a linear time-invariant (LTI) system

$$\dot{x} = Ax + B_1 d + B_2 u \tag{3}$$

is to find a controller that for the worst-case disturbance ($d \in R^q$) minimizes the following linear-quadratic performance functional

$$J(u,d) = \int_0^\infty \left(x^T Q x + u^T R u - \gamma^2 d^T d \right) dt \tag{4}$$

where $Q = Q^T > 0$ and $R = R^T > 0$ are matrices, and γ is the disturbance attenuation level that is usually a prescribed value.

For the minimization of the performance functional J, we are assured that the required unique positive-definite solution $P = P^T > 0$ of the generalized algebraic Riccati equation (GARE)

$$0 = PA + A^T P + P\left(\frac{1}{\gamma^2} B_1 B_1^T - B_2 R^{-1} B_2^T \right) P + Q \tag{5}$$

exists if (A, B_1) is controllable and (A, Q) is observable [35].

(2) Nonlinear System: It is known [34-35] that for the following type of general nonlinear system

$$\dot{x} = f(x) + g_1(x)d + g_2(x)u \tag{6}$$

where $x \in R^n$ is the n-dimensional state, $u \in R^p$ is the control input, and $d \in R^q$ is an unknown disturbance signal. Suppose that there exists a positive optimal value function $V(x)$, which satisfies the following HJI equation

$$\frac{\partial V}{\partial t} + L_f V - \frac{1}{4}\left(L_{g_2} V\right) r(x)^{-1} \left(L_{g_2} V\right)^T + \frac{1}{4}\gamma^{-2}\left(L_{g_1} V\right)\left(L_{g_1} V\right)^T + q(x) = 0 \tag{7}$$

Then

$$u^*(x) = -\frac{1}{2} r(x)^{-1} \left(L_{g_2} V\right)^T \tag{8}$$

is the optimal stabilizing control which minimizes the performance functional

$$J(u,d) = \int_0^\infty \left(q(x) + u^T r(x) u - \gamma^2 d^T d\right) dt \tag{9}$$

where $q(x)$ is positive definite and radially unbounded, $r(x)$ is also positive definite, and $\gamma^2 d^T d$ represents the level of disturbance attenuation. In addition, worst case disturbance $d^*(x)$ is

$$d^*(x) = \frac{1}{2\gamma^2}\left(L_{g_1} V\right)^T \tag{10}$$

That is

$$\min_{u^*} \max_{d^*} \left(q(x) + u^T r(x) u - \gamma^2 d^T d + \dot{V}(x)\right) = 0 \tag{11}$$

Remark 2.2: As discussed in the book of [35], this nonlinear H_∞ problem is equivalent to the differential game $\min_u \max_d J(u,d)$ for the performance functional (9) constrained by nonlinear system (6). Because it is too difficult to solve the formidable HJI equation of (7) in general, finding a nonlinear H_∞ optimal control for the general nonlinear system (6) still remains open.

Remark 2.3: It should be pointed out that the control law represented by (8) achieves stability margins which are a nonlinear analog of Kalman's margins for the linear quadratic regulators, that is, the infinite gain margin and the 60° phase margins (See [40] for details).

3. Main Results

Prior to developing a nonlinear H_∞ optimal control for delayed recurrent neural networks influenced by uncertain noise, we first introduce the following result, which achieves a globally stabilizing control that guarantees the system of (2) is input-to-state stability.

Theorem 1: For the noisy time-delay recurrent neural networks modeled by (2), if we choose the control as

$$u = -\left(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + 2l^2\right)x(t) \tag{12}$$

in which both $h = \max\{h_j\}$ and $k = \max\{k_j\}$ for all j $(j = 1, \cdots, n)$, and l is a design parameter, then the system of (2) achieves input-to-state stability.

Proof: Consider the following positive-definite Lyapunov function

$$V = \frac{1}{2}x(t)^T x(t) + \int_{t-\tau}^{t} \left((W_2 g(x(s)))^T (W_2 g(x(s)))\right) ds \tag{13}$$

Taking the derivative of it along the trajectory of (2), we obtain

$$\dot{V} = x(t)^T \left(-Ax(t) + W_1 f(x(t)) + W_2 g(x(t-\tau)) + u + \Psi\right) + (W_2 g(x(t)))^T (W_2 g(x(t)))$$
$$- (W_2 g(x(t-\tau)))^T (W_2 g(x(t-\tau)))$$

$$=-\lambda x(t)^T x(t)+x(t)^T W_1 f(x(t))+x(t)^T W_2 g(x(t-\tau))+x(t)^T u+x(t)^T \Psi$$

$$+(W_2 g(x(t)))^T (W_2 g(x(t)))-(W_2 g(x(t-\tau)))^T (W_2 g(x(t-\tau))) \tag{14}$$

Let us apply the following Young's Inequality to both the second term $x(t)^T W_1 f(x(t))$ and the third term $x(t)^T W_2 g(x(t(t-\tau))$ in (14),

$$x^T y \le \frac{\|x\|^2}{2}+\frac{\|y\|^2}{2}=\frac{1}{2}x^T x+\frac{1}{2}y^T y \tag{15}$$

in which *x* and *y* are two vectors.

We acquire

$$x(t)^T W_1 f(x(t)) \le \frac{1}{2}x(t)^T x(t)+\frac{1}{2}\|W_1 f(x(t))\|^2$$

$$\le \frac{1}{2}x(t)^T x(t)+\frac{1}{2}\|W_1\|^2 \|f(x(t))\|^2 \tag{16}$$

and

$$x(t)^T W_2 g(x(t-\tau)) \le \frac{1}{2}x(t)^T x(t)+\frac{1}{2}(W_2 g(x(x-\tau)))^T (W_2 g(x(x-\tau))) \tag{17}$$

From *Definition 2.2*, we have

$$\left|f_j(x_j)\right| \le h_j\left|x_j\right|, \forall x_j \in R,\ j=1,2,\cdots,n$$

Therefore

$$\|f(x(t))\|^2 \le (Hx(t))^T (Hx(t))=x(t)^T H^2 x(t) \le h^2 x(t)^T x(t) \tag{18}$$

where $H=diag(h_1,h_2,\cdots,h_n)$ and $h=\max\{h_j\},\ j=1,\cdots,n$.

Then from (16), we gain

$$x(t)^T W_1 f(x(t)) \le \frac{1}{2}x(t)^T x(t)+\frac{1}{2}\|W_1\|^2 h^2 x(t)^T x(t)=x(t)^T\left(\frac{1+\|W_1\|^2 h^2}{2}\right)x(t) \tag{19}$$

By applying Young's Inequality (15) to the fifth term in (14) again, we get

$$x(t)^T \Psi \le l^2 \|x(t)\|^2 + \frac{1}{4l^2}\|\Psi\|^2$$

$$= l^2 x(t)^T x(t) + \frac{1}{4l^2}\Psi^T \Psi \tag{20}$$

where l is a design parameter.

In addition, with respect to the sixth term in (14), we have

$$(W_2 g(x(t)))^T (W_2 g(x(t))) = \|W_2 g(x(t))\|^2 \le \|W_2\|^2 \|g(x(t))\|^2 \tag{21}$$

From *Definition 2.2*, we have

$$\left|g_j(x_j)\right| \le k_j \left|x_j\right|, \forall x_j \in R,\ j = 1,2,\cdots,n$$

and

$$\|g(x(t))\|^2 \le (Kx(t))^T (Kx(t)) = x(t)^T K^2 x(t) \le k^2 x(t)^T x(t) \tag{22}$$

where $K = diag(k_1, k_2, \cdots, k_n)$ and $k = \max\{k_j\},\ j = 1,\cdots,n$.

Therefore, combining (21) and (22) together, we have

$$(W_2 g(x(t)))^T (W_2 g(x(t))) \le \|W_2\|^2 \|g(x(t))\|^2 \le \|W_2\|^2 k^2 x(t)^T x(t) \tag{23}$$

Substitute (17), (19), (20), and (23) into (14), we reach

$$\dot{V} \le -\lambda x(t)^T x(t) + x(t)^T \left(\frac{1 + \|W_1\|^2 h^2}{2} \right) x(t) + \frac{1}{2} x(t)^T x(t) + \frac{1}{2} (W_2 g(x(x-\tau)))^T (W_2 g(x(x-\tau)))$$

$$+ x(t)^T u + l^2 x(t)^T x(t) + \frac{1}{4l^2}\Psi^T \Psi + \|W_2\|^2 k^2 x(t)^T x(t) - (W_2 g(x(t-\tau)))^T (W_2 g(x(t-\tau)))$$

$$= -\lambda x(t)^T x(t) + x(t)^T \left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + 2l^2}{2} \right) x(t) + x(t)^T u + \frac{1}{4l^2}\Psi^T \Psi$$

$$-\frac{1}{2}(W_2 g(x(t-\tau)))^T (W_2 g(x(t-\tau))) \tag{24}$$

Because of $(W_2 g(x(t-\tau)))^T (W_2 g(x(t-\tau))) \geq 0$,

we finally achieve

$$\dot{V} \leq -\lambda x(t)^T x(t) + x(t)^T \left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + 2l^2}{2} \right) x(t) + x(t)^T u + \frac{1}{4l^2} \Psi^T \Psi \tag{25}$$

With the control of (12), the inequality above becomes

$$\dot{V} \leq -\lambda x(t)^T x(t) + \frac{1}{4l^2} \Psi^T \Psi - x(t)^T \left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + 2l^2}{2} \right) x(t) \tag{26}$$

Let us define

$$r(|x|) = x(t)^T \left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + 2l^2}{2} \right) x(t) = \left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + 2l^2}{2} \right) \|x(t)\|^2 \tag{27}$$

Thus, we get

$$\dot{V} \leq -x(t)^T \left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + 2l^2}{2} \right) x(t) = -r(|x|) \tag{28}$$

whenever $\|x(t)\| \geq \dfrac{\|\Psi\|}{2l\sqrt{\lambda}}$.

By the definition of input-to-state stability (in [40], page 13, Definition 2.1) and Theorem 2.2 (Sontag) of the reference [40] (in [40], page 14), we conclude that the system of (2) is input-to-state stability with the control of (12). This completes the proof.

Remark 3.1: Although the Lyapunov theory, the most successful and widely used tool to determine the stability of a system, was invented a century ago, there are still no systematic methods to obtain Lyapunov functions for general nonlinear systems. Therefore, it remains a challenging task to find a Lyapunov function and stabilize a nonlinear system [41].

Now, let us define

$$\gamma^2 = \frac{1}{4l^2} \;\; \Rightarrow \;\; l^2 = \frac{1}{4\gamma^2} \tag{29}$$

Then, the control (12) becomes

$$u = -2\left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2}{2} + \frac{1}{4\gamma^2} \right) x(t) \tag{30}$$

Let us consider the Lyapunov function V as the optimal value function and substitute it into (7), which yields the next equation

$$W_2 g(x(t)))^T (W_2 g(x(t))) - (W_2 g(x(t-\tau)))^T (W_2 g(x(t-\tau))) - \lambda x(t)^T x(t) + x(t)^T W_1 f(x(t)) + x(t)^T W_2 g(x(t-\tau))$$

$$-\frac{1}{4} x(t)^T r(x(t))^{-1} x(t) + \frac{1}{4} \gamma^{-2} x(t)^T x(t) + q(x(t)) = 0 \tag{31}$$

Let us select a new control that is a modification of (30)

$$u = -c\left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2}{2} + \frac{1}{4c\gamma^2} \right) x(t)$$

$$= -\frac{1}{2} c\left(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + \frac{1}{2c\gamma^2} \right) x(t) \tag{32}$$

where $c > 2$ is a constant.

We then choose the function $r(x(t))$ as

$$r(x(t)) = c^{-1}(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + \frac{1}{2c\gamma^2})^{-1} \tag{33}$$

and from (31) the function $q(x(t))$ is given by

$$q(x(t)) = \lambda x(t)^T x(t) + \frac{c}{4}(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + \frac{1}{2c\gamma^2}) x(t)^T x(t) - \frac{1}{4}\gamma^{-2} x(t)^T x(t) - x(t)^T W_1 f(x(t))$$

$$- x(t)^T W_2 g(x(t-\tau)) + (W_2 g(x(t-\tau)))^T (W_2 g(x(t-\tau))) - (W_2 g(x(t)))^T (W_2 g(x(t))) \tag{34}$$

We now have the following theorem.

Theorem 2: For the noisy time-delay recurrent neural networks described by (2), there exist a positive-definite function $q(x(t))$ (34) and a strictly positive function $r(x(t))$ (33), such that the feedback control law

$$u = u^* = -\frac{1}{2} r(x(t))^{-1} x(t) \tag{35}$$

achieves an optimal input-to-state stabilization with respect to a meaningful performance functional

$$J = \int_0^\infty \left(q(x) + u^T r(x) u - \gamma^2 \Psi^T \Psi \right) dt \tag{36}$$

for the worst case Ψ

$$\Psi = \Psi^* = \frac{1}{2\gamma^2} x(t) \tag{37}$$

In addition, $x(t) \to 0$ as $t \to \infty$ for all $\Psi \in L_2$, and $x(t) \in L_\infty$ for all $\Psi \in L_\infty$. Without a disturbance, $\Psi = 0$, the control law (31) makes the equilibrium $x = 0$ global asymptotic stability (GAS).

Proof:

Step 1: By considering the Lyapunov function candidate V represented by (13), the derivative of V along the trajectory of (2) is

$$\begin{aligned} \dot{V} &= -\lambda x(t)^T x(t) + x(t)^T W_1 f(x(t)) + x(t)^T W_2 g(x(t-\tau)) + x(t)^T u + x(t)^T \Psi \\ &\quad + (W_2 g(x(t)))^T (W_2 g(x(t))) - (W_2 g(x(t-\tau)))^T (W_2 g(x(t-\tau))) \end{aligned} \tag{38}$$

The substitution of the control law (35) into $\dot{V}$ yields

$$\begin{aligned} \dot{V} &= -\lambda x(t)^T x(t) + x(t)^T W_1 f(x(t)) + x(t)^T W_2 g(x(t-\tau)) - x(t)^T \left(\frac{1}{2} c \left(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + \frac{1}{2c\gamma^2} \right) x(t) \right) \\ &\quad + x(t)\Psi + (W_2 g(x(t)))^T (W_2 g(x(t))) - (W_2 g(x(t-\tau)))^T (W_2 g(x(t-\tau))) \end{aligned} \tag{39}$$

From (17), (19), (20), and (23), we obtain

$$\dot{V} \le -\lambda x(t)^T x(t) + x(t)^T \left(\frac{1+\|W_1\|^2 h^2}{2} \right) x(t) + \frac{1}{2} x(t)^T x(t) + \frac{1}{2} (W_2 g(x(x-\tau)))^T (W_2 g(x(x-\tau)))$$

$$- x(t)^T \left(\frac{1}{2} c \left(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + \frac{1}{2c\gamma^2} \right) x(t) \right) + l^2 x(t)^T x(t) + \frac{1}{4l^2} \Psi^T \Psi$$

$$+ (W_2 g(x(t)))^T (W_2 g(x(t))) - (W_2 g(x(t-\tau)))^T (W_2 g(x(t-\tau)))$$

$$\le -\lambda x(t)^T x(t) - \left(\left(\frac{c-1}{2} \right) \left(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 \right) \right) x(t)^T x(t) + \gamma^2 \Psi^T \Psi$$

$$- \frac{1}{2} (W_2 g(x(t-\tau)))^T (W_2 g(x(t-\tau)))$$

$$\le -\lambda x^T x + \gamma^2 \Psi^T \Psi - \left(\left(\frac{c-1}{2} \right) \left(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 \right) \right) x(t)^T x(t) \qquad (40)$$

Let us define

$$r(|x|) = \left(\left(\frac{c-1}{2} \right) \left(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 \right) \right) x(t)^T x(t) = \left(\left(\frac{c-1}{2} \right) \left(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 \right) \right) \|x(t)\|^2 \quad (41)$$

Therefore $\dot{V} \le -r(|x|)$ whenever $\|x\| \ge \frac{\gamma}{\sqrt{\lambda}} \|\Psi\|$.

Same as Theorem 1, we conclude that the system of (2) achieves input-to-state stabilization with the control law (35). In addition, By [34] and [42], we know that $x(t) \to 0$ as $t \to \infty$ for all $\Psi \in L_2$, and $x(t) \in L_\infty$ for all $\Psi \in L_\infty$. In the absence of a disturbance, $\Psi = 0$, the control law (35) renders the equilibrium $x = 0$ global asymptotic stability (GAS).

Step 2: Let us consider $q(x(t))$ and $r(x(t))$

From (34)

$$q(x(t)) = \lambda x(t)^T x(t) + \frac{c}{4}(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + \frac{1}{2c\gamma^2}) x(t)^T x(t) - \frac{1}{4} \gamma^{-2} x(t)^T x(t) - x(t)^T W_1 f(x(t))$$

$$- x(t)^T W_2 g(x(t-\tau)) + (W_2 g(x(t-\tau)))^T (W_2 g(x(t-\tau))) - (W_2 g(x(t)))^T (W_2 g(x(t))) \qquad (42)$$

Substitute (17), (19), (20), and (23) into (42) above, we get

$$q(x(t)) \geq \lambda x(t)^T x(t) + \frac{c}{4}(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + \frac{1}{2c\gamma^2})x(t)^T x(t) - \frac{1}{4}\gamma^{-2} x(t)^T x(t) - x(t)^T \left(\frac{1 + \|W_1\|^2 h^2}{2} \right) x(t)$$

$$- \frac{1}{2} x(t)^T x(t) - \frac{1}{2}(W_2 g(x(x-\tau)))^T (W_2 g(x(x-\tau)))$$

$$+ (W_2 g(x(t-\tau)))^T (W_2 g(x(t-\tau))) - \|W_2\|^2 k^2 x(t)^T x(t)$$

$$= \lambda x(t)^T x(t) + \left(\frac{c-2}{2} \right) \left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2}{2} \right) x(t)^T x(t) - \frac{1}{8}\gamma^{-2} x(t)^T x(t)$$

$$+ \frac{1}{2}(W_2 g(x(t-\tau)))^T (W_2 g(x(t-\tau)))$$

$$\geq \left(\lambda - \frac{1}{8}\gamma^{-2} \right) x(t)^T x(t) + \left(\frac{c-2}{2} \right) \left(\frac{2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2}{2} \right) x(t)^T x(t)$$

$$\geq 0 \tag{43}$$

Then, $q(x(t))$ is positive definite and radially unbounded.

By (33)

$$r(x(t)) = c^{-1}(2 + \|W_1\|^2 h^2 + 2\|W_2\|^2 k^2 + \frac{1}{2c\gamma^2})^{-1} \tag{44}$$

it is obvious that $r(x(t)) > 0$.

With the choice of $q(x(t))$ and $r(x(t))$ in (34) and (33), $\dot{V}$ can be written into the following form

$$\dot{V} = -q(x(t)) - u^T R(x(t)) u + \gamma^2 \Psi^T \Psi - \gamma^2 \left\| \frac{1}{2\gamma^2} x(t) - \Psi \right\|^2 + (u - u^*)^T r(x(t))(u - u^*) \tag{45}$$

From the equation (45) above, we know that the optimal control $u = u^*$ is an optimal solution to J (36) for the worst disturbance noise $\Psi = \Psi^* = \frac{1}{2\gamma^2} x(t)$, that is,

$$\min_{u^*} \max_{\Psi^*} \left(q(x) + u^T r(x) u - \gamma^2 \Psi^T \Psi + \dot{V}(x) \right) = 0 \tag{46}$$

In addition, the Lyapunov function V (13) satisfies the HJI equation

$$\frac{\partial V}{\partial t} + L_f V - \frac{1}{4}\left(L_{g_2} V\right) r(x)^{-1} \left(L_{g_2} V\right)^T + \frac{1}{4}\gamma^{-2}\left(L_{g_1} V\right)\left(L_{g_1} V\right)^T + q(x) = 0 \tag{47}$$

Therefore, V is the value function for the performance functional (36). This completes the proof.

Remark 3.2: The relationship between stability and optimality has been a central issue in nonlinear systems analysis [43]. In our paper, the control law (35) achieves both input-to-state stability and optimality with respect to a meaningful performance functional. Because of our inability to solve the Hamilton-Jacobi-Isaacs equation directly, the formulation of H_∞ control in this paper was developed with the help of an alternative method, inverse optimality, which has been shown that it appears to be one of the most promising methods in the area of modeling biologically-inspired neural networks [44]. It is our hope to find a direct nonlinear optimal control in the future.

Remark 3.3: A real system is usually affected by both internal parametric uncertainties and external noise perturbations. They are different, and cannot be transformed between each other. According to the formulation of noisy recurrent neural networks given by [38], we focused on the robust stability analysis problem for delayed recurrent neural networks influenced by uncertain noise perturbations, which are modeled by a deterministic system (2). With respect to parametric uncertainties, robust adaptive control could be an effective methodology, which will be our future work.

4. Numerical Examples

In this section, we will give two examples to verify the theoretical analysis and demonstrate the effectiveness of our approach. Two numerical examples are simulated by using the Matlab software as well as the fifth-order Dormand-Prince method with a fixed step size of 0.005.

Example 1: A noisy time-delay recurrent neural network is given as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2.11 & -0.12 \\ -5.09 & 3.18 \end{bmatrix}\begin{bmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{bmatrix} + \begin{bmatrix} -1.59 & -0.11 \\ -0.19 & -2.47 \end{bmatrix}\begin{bmatrix} \tanh(x_1(t-\tau)) \\ \tanh(x_2(t-\tau)) \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \tag{48}$$

where $x_1(0) = 0.3$, $x_2(0) = -3$, $\lambda = -1$, $W_1 = \begin{bmatrix} 2.11 & -0.12 \\ -5.09 & 3.18 \end{bmatrix}$, $W_2 = \begin{bmatrix} -1.59 & -0.11 \\ -0.19 & -2.47 \end{bmatrix}$, activation function $f_j(x_j) = g_j(x_j) = \tanh(x_j)$ ($j = 1, 2$), $\tau = 1$, and Ψ_1, Ψ_2 are uniformly random noises with the magnitude of $\left|\Psi_j\right| = 1$ ($j = 1, 2$).

Fig. 1 shows the result of phase plane of the neural network without both the noise and the control. One can observe that this network is a chaotic delayed recurrent neural network, which means that it does not have the desired property of global asymptotic stability. Fig. 2 and 3 display the results of phase plane and time response of the neural network with the noise ($\left|\Psi_j\right| = 1$) but without the control. Once again, the system is unstable and chaotic. The effect of the noise in the system can be seen obviously, which further worsen the system. Finally, at $t = 50$, let us apply the H_∞ control signal (35) to the system influenced by the noise. The parameters of the controller are chosen as follows: $c = 4$, $h = k = 1$, and $\gamma = 1/2$. The result is given by Fig. 4. One can see that the neural network is globally asymptotically stable, that is, it achieves the input-to-state stability.

Example 2: Let us consider the following noisy time-delay recurrent neural network

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} f_1(x_1(t)) \\ f_2(x_2(t)) \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} g_1(x_1(t-\tau)) \\ g_2(x_2(t-\tau)) \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \tag{49}$$

where $x_1(0) = 5$, $x_2(0) = -5$, $\lambda = -1$, $W_1 = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, $W_2 = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, activation function $f_j(x_j) = g_j(x_j) = \dfrac{1}{1+\exp(-2x_j)} - 0.5$ ($j = 1, 2$), $\tau = 1$, and Ψ_1, Ψ_2 are uniformly random noises with the magnitude of $\left|\Psi_j\right| = 10$ ($j = 1, 2$).

Fig. 5 represents the result of phase plane of the neural network without both the noise and the control. From the picture, it is obvious that the network is not globally asymptotically stable. Fig. 6 and 7 display the results of phase plane and time response of the neural network with the noise ($\left|\Psi_j\right| = 10$) but without the control. The system is unstable and becomes chaotic. Finally, at $t = 50$, the proposed H_∞ control signal (35) is inputted at $t = 50$ to the system. The parameters of the controller are chosen as follows: c = 16, h = k = 0.5, and $\gamma = \frac{1}{2}$. From the results of Fig. 8, one can observe that the system achieves the expected performance, i.e., input-to-state stability.

Example 2: Using recurrent neural networks has become one of the most promising methods in the identification and control of unknown nonlinear systems. The methodologies with which recurrent neural networks are able to identify nonlinear affine systems have been developed in [45]. The idea is to substitute an unknown nonlinear system by a recurrent neural network structure. Therefore, the controller can be directly designed based on the model of recurrent neural network to control the nonlinear system indirectly [45]. Let us consider the following first-order nonlinear system

$$\dot{y} = -\sin y + u \tag{50}$$

where $y(0)$ = 4. Before we proceed any further, it is important to note that (50) is used in simulation only to generate the signal for measurement. The actual controller does not take into account the system dynamics. Although this nonlinear system is simple, it is interesting nevertheless, that is, when u = 0, $\mathrm{y}(\infty) = k\pi$, k = 0,1,2,…. Therefore, the system is not global asymptotic stability. To stabilize the unknown system, we select a one-dimensional recurrent neural network as follows:

$$\dot{x} = -0.5x + W_1 \tanh(x) + u \tag{51}$$

in which $x(o) = 4$. It is a deterministic neural network without time delay. Thus $W_2 = 0$. As in [45], we use the following on-line learning algorithm:

$$\dot{W}_1 = -K\varDelta_x \tanh(x) \tag{52}$$

where $K = 10$, $\Delta_x = x - y$, and $W_1(0) = -0.5$.

The simulation results show that our proposed approach can be applied to control an unknown nonlinear system. Fig. 13 shows the time evolution of the system state y when u = 0. Fig. 14 shows the response of the system state y when the proposed control signal (32) is applied. The parameters of the controller are as follows: c = 4, L = 1, H = 0, and $\| M \| = 0$. After the control, the system (50) becomes global asymptotic stability.

Remark 4.1: To show the effectiveness of the proposed approach broadly, two neural networks (Example 1 and Example 2) above have different activation functions. Example 1 has the activation function $\tanh(x_j)$ and Example 2 has the activation function $\frac{1}{1+\exp(-2x_j)} - 0.5$. To consider all situations from various perspectives, a low level of disturbance noises ($\left|\Psi_j\right| = 1$) is applied to Example 1 and a high level of disturbance noises ($\left|\Psi_j\right| = 10$) is applied to Example 2.

5. Conclusions

This paper has presented a new theoretical design of a nonlinear H_∞ optimal control for delayed recurrent neural networks with noise uncertainty to achieve the input-to-state stability optimally. The formulation of H_∞ control is developed by using both Lyapunov technique and the concept of inverse optimality, and solving a Hamilton-Jacobi-Isaacs (HJI) equation indirectly. Therefore, the closed-loop feedback system is able to attenuate noises to a prescribed level with stability margins. Two globally stabilizing control laws have been derived and two numerical examples have been presented to demonstrate the effectiveness of the aforementioned approach. Simulation results show that the proposed approach turns out to be very effective for different systems under different circumstances. It is believed that the new design presented in this paper would accelerate the applications of recurrent neural networks.

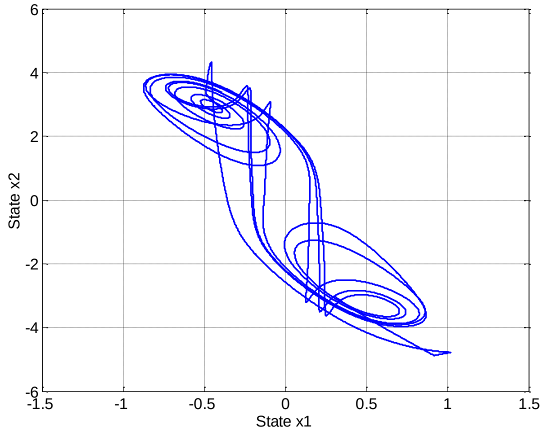


Fig. 1. Phase Plane (u = 0 and $\Psi_j = 0$ (j = 1, 2))

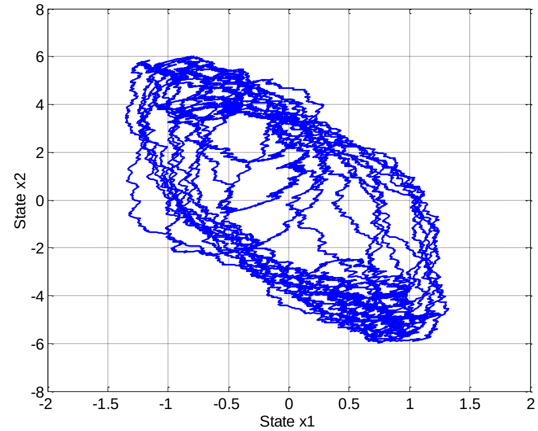


Fig. 2. Phase Plane (u = 0 and $|\Psi_j| = 1$ (j = 1, 2))

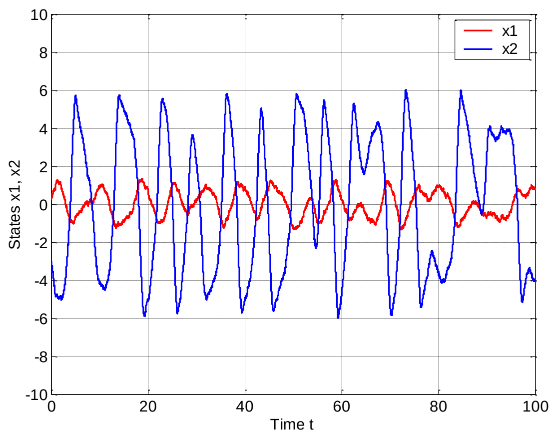


Fig. 3. System Response without the Control

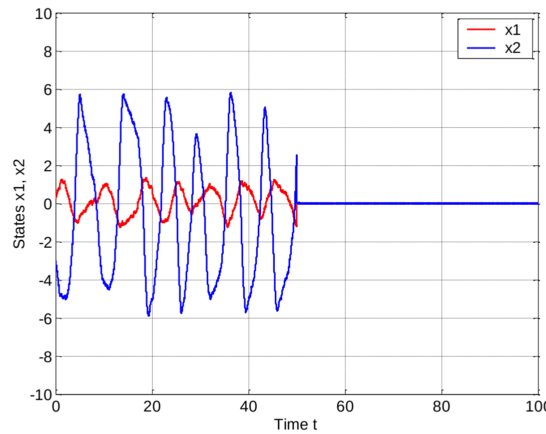


Fig. 4. System Response with the Control (35) at $t = 50$

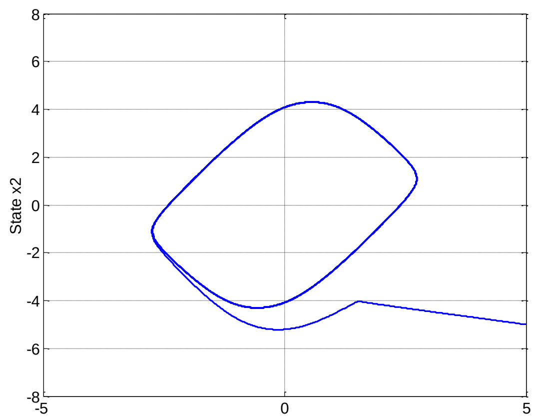


Fig. 5. Phase Plane (u = 0 and $\Psi_j = 0$ (j = 1, 2))

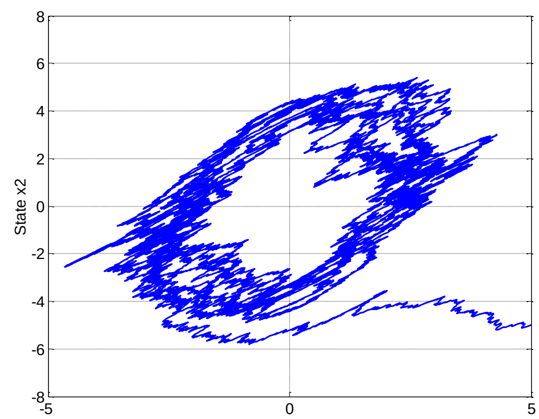
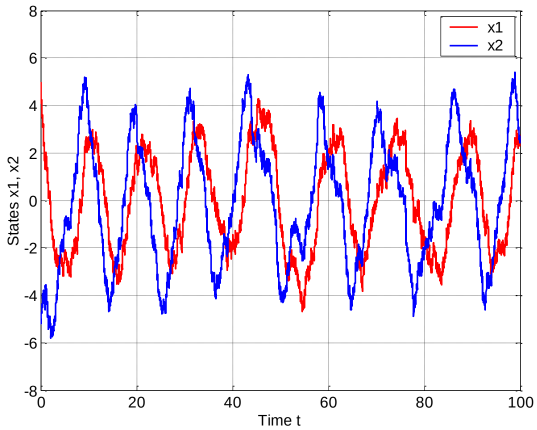


Fig. 6. Phase Plane (u = 0 and $|\Psi_j| = 10$ (j = 1, 2))



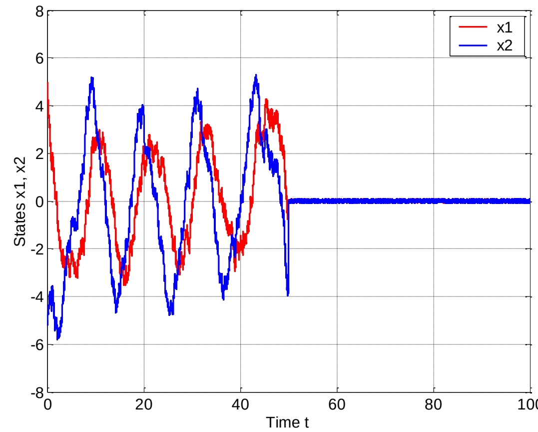


Fig. 7. System Response without the Control

Fig. 8. System Response with the Control (35) at $t = 50$

References:

[1] Z. Tu, J. Jian, K. Wang, Global exponential stability in Lagrange sense for recurrent neural networks with both time-varying delays and general activation functions via LMI approach, Nonlinear Analysis: Real World Applications. 12 (2011) 2174–2182.

[2] X. Li, X. Fu, P. Balasubramaniam, R. Rakkiyappan, Existence, uniqueness and stability analysis of recurrent neural networks with time delay in the leakage term under impulsive perturbations, Nonlinear Analysis: Real World Applications. 11 (2010) 4092–4108.

[3] J. Yu, K. Zhang, S. Fei, Exponential stability criteria for discrete-time recurrent neural networks with time-varying delay, Nonlinear Analysis: Real World Applications. 11 (2010) 207–216.

[4] T. Li, Q. Luo, C. Sun, B. Zhang, Exponential stability of recurrent neural networks with time-varying discrete and distributed delays, Nonlinear Analysis: Real World Applications. 10 (2009) 2581–2589.

[5] Y. Guo, New results on input-to-state convergence for recurrent neural networks with variable inputs, Nonlinear Analysis: Real World Applications. 9 (2008) 1558–1566.

[6] B. Liu, L. Huang, Positive almost periodic solutions for recurrent neural networks, Nonlinear Analysis: Real World Applications. 9 (2008) 830–841.

[7] X. Liao, Q. Luo, Z. Zeng, Y. Guo, Global exponential stability in Lagrange sense for recurrent neural networks with time delays, Nonlinear Analysis: Real World Applications. 9 (2008) 1535–1557.

[8] J. Liang, J. Cao, Global output convergence of recurrent neural networks with distributed delays, Nonlinear Analysis: Real World Applications. 8 (2007) 187–197.

[9] Q. Song, J. Cao, Z. Zhao, Periodic solutions and its exponential stability of reaction–diffusion recurrent neural networks with continuously distributed delays, Nonlinear Analysis: Real World Applications. 7 (2006) 65–80.

[10] M. S. Mahmoud, Y. Xia, Improved exponential stability analysis for delayed recurrent neural networks, Journal of the Franklin Institute. 348 (2011) 201–211.

[11] M. U. Akhmet, D. Arugaslan, E. Yilmaz, Stability analysis of recurrent neural networks with piecewise constant argument of generalized type, Neural Networks. 23 (2010) 805–811.

[12] L. Wang, R. Zhang, Z. Xu, J. Peng, Some characterizations of global exponential stability of a generic class of continuous-time recurrent neural networks, IEEE Transactions on Systems, Man, and Cybernetics – Part B: Cybernetics. 39 (2009) 763–772.

[13] H. Shao, Delay-dependent approaches to globally exponential stability for recurrent neural networks, IEEE Transactions on Circuits and Systems II: Express Briefs. 55 (2008) 591–595.

[14] H. Zhang, Z. Wang, D. Liu, Robust exponential stability of recurrent neural networks with multiple time-varying delays, IEEE Transactions on Circuits and Systems II: Express Briefs. 54 (2007) 730–734.

[15] S. Arik, Global asymptotic stability analysis of bidirectional associative memory neural networks with time delays, IEEE Transactions on Neural Networks. 16 (2005) 580–586.

[16] Z. Wang, Y. Liu, L. Yu, X. Liu, Exponential stability of delayed recurrent neural networks with Markovian jumping parameters, Physics Letters A. 356 (2006) 346–352.

[17] J. Cao, D. Huang, Y. Qu, Global robust stability of delayed recurrent neural networks, Chaos, Solitons and Fractals. 23 (2005) 221–229.

[18] V. Singh, A generalized LMI-based approach to the global asymptotic stability of delayed cellular neural networks, IEEE Transactions on Neural Networks. 15 (2004) 223–225.

[19] J. Lian, Z. Feng, P. Shi, Observer design for switched recurrent neural networks: an average dwell time approach, IEEE Transactions on Neural Networks. 22 (2011) 1547–1556.

[20] J. Lian, K. Zhang, Exponential stability analysis for switched Cohen-Grossberg neural networks with average dwell time, Nonlinear Dynamics. 63 (2011) 331–343.

[21] X Su, Z Li, Y Feng, L Wu, New global exponential stability criteria for interval delayed neural networks, Proceedings of the Institution of Mechanical Engineers - Part I: Journal of Systems and Control Engineering. 225 (2011) 125–136.

[22] X. Dong, Y. Zhao, Y. Xu, Z. Zhang, P. Shi, Design of PSO fuzzy neural network control for ball and plate system, International Journal of Innovative Computing, Information and Control. 7 (2011) 7091–7103.

[23] C. K. Ahn, M. K. Song, L_2 -L_∞ Filtering for time-delayed switched Hopfield neural networks, International Journal of Innovative Computing, Information and Control. 7 (2011) 1831–1843.

[24] I. Ahmad, A. Abdullah, A. Alghamdi, Investigating supervised neural networks to intrusion detection, ICIC Express Letters. 4 (2010) 2133–2138.

[25] Y. E. Shao, An integrated neural networks and SPC approach to identify the starting time of a process disturbance, ICIC Express Letters. 3 (2009) 319–324.

[26] R. Yang, Z. Zhang, P. Shi, Exponential stability on stochastic neural networks with discrete interval and distributed delays, IEEE Transactions on Neural Networks. 21 (2010) 169–175.

[27] L. Wan, Q. Zhou, Attractor and ultimate boundedness for stochastic cellular neural networks with delays, Nonlinear Analysis: Real World Applications. 12 (2011) 2561–2566.

[28] R. Sakthivel, R. Samidurai, S. M. Anthoni, Asymptotic stability of stochastic delayed recurrent neural networks with impulsive effects, Journal of Optimization Theory and Applications. 147 (2010) 583–596.

[29] Y. Lv, W. Lv, J. Sun, Convergence dynamics of stochastic reaction–diffusion recurrent neural networks with continuously distributed delays, Nonlinear Analysis: Real World Applications. 9 (2008) 1590–1606.

[30] Z. Liu, H. Schurz, N. Ansari, and Q. Wang, Theoretic design of differential minimax controllers for stochastic cellular neural networks, Neural Networks, 26 (2012) 110–117.

[31] S. K. Nguang, P. Shi, H_∞ Fuzzy output feedback control design for nonlinear systems: an LMI approach, IEEE Transactions on Fuzzy Systems. 11 (2003) 331–340.

[32] S. K. Nguang, P. Shi, Nonlinear H_∞ filtering of sampled data systems, Automatica. 36 (2000) 303–310.

[33] L. Wu, X. Su, P. Shi, Mixed H_2/H_∞ approach to fault detection of discrete linear repetitive processes, Journal of the Franklin Institute. 348 (2011) 393–414.

[34] K. Ezal, Z. Pan, P. V. Kokotovic, Locally optimal and robust backstepping design, IEEE Transactions on Automatic Control. 45 (2000) 260–271.

[35] T. Basar, P. Bernhard, H_∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach, 2nd Edition, Birkhauser, Boston, MA, 1995.

[36] C. Lu, W. Shyr, K. Yao, C. Liao, C. Huang, Delay-dependent H_∞ control for discrete-time uncertain recurrent neural networks with interval time-varying delay, International Journal of Innovative Computing, Information and Control. 5 (2009) 3483–3493.

[37] W. Yu, J. Cao, Robust control of uncertain stochastic recurrent neural networks with time-varying delay, Neural Processing Letters. 26 (2007) 101–119.

[38] S. Das, O. Olurotimi, Noisy recurrent neural networks: the continuous-time case, IEEE Transactions on Neural Networks. 9 (1998) 913–936.

[39] J. Cao, J. Wang, Global asymptotic stability of a general class of recurrent neural networks with time-varying delays, IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications. 50 (2003) 34–44.

[40] M. Krstic, H. Deng, Stabilization of Nonlinear Uncertain Systems, Springer-Verlag, New York, 1998.

[41] J. Primbs, V. Nevistic, J. Doyle, Nonlinear optimal control: a control Lyapunov function and receding horizon perspective, Asian Journal of Control. 1 (1999) 14–24.

[42] A. Teel, Asymptotic convergence from L_p Stability, IEEE Transactions on Automatic Control. 44 (1999) 2169–2170.

[43] R. Freeman, P. Kokotovic, Robust Control of Nonlinear Systems, Birkhauser, Boston, MA, 1996.

[44] E. Todorov, Optimal control theory, in: K. Doya et al (Eds), Bayesian Brain: Probabilistic Approaches to Neural Coding, MIT Press, Massachusetts, 2006, pp. 269–298.

[45] G. Rovitahkis and M. Christodoulou, Adaptive Control with Recurrent High-order Neural Networks, Springer-Verlag, New York, 2000.