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An exponential observer design for the unified Rossler chaotic system

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Abstract

In this paper, the unified Rossler chaotic system is addressed and the state observation problem of such a system is explored. Based on the time-domain approach with differential and integral inequalities, a suitable state observer for the unified Rossler chaotic system is established to assure the global exponential stability of the resulting error system. Besides, the guaranteed exponential decay rate can be accurately estimated. Finally, numerical simulations are offered to demonstrate the feasibility and effectiveness of the obtained results.

KeyWords: Unified Rossler chaotic system; Observer design; Chaotic system; Exponential decay rate

1. Introduction

In recent years, numerous chaotic systems have been extensively explored; see, for example, [1-10] and the references therein. As a rule, chaos in many dynamic systems is a factor of the generation of oscillation and a factor of instability. Chaos commonly occurred in various fields of application; for instance, secure communication, system identification, and ecological systems. Based on the practical considerations, it is either inappropriate or impossible to measure all the elements of the state vector. Moreover, a state observer can be used to replace sensor signals, in the event of sensor failures. It goes without saying that the state observer design of physical systems with chaos is not as easy as that without chaos. Owing to the above-mentioned reasons, the observer design of chaotic systems is really prerequisite and significant.

In this paper, the observability problem for the unified Rossler chaotic system is explored. By using the time-domain approach with differential and integral inequalities, a suitable state observer for the unified Rossler chaotic system is offered to ensure the global exponential stability of the resulting error system. Meanwhile, the guaranteed exponential decay rate can be precisely calculated. Finally, some numerical simulations are given to show the effectiveness of the obtained result.

2. Problem formulation and main results

Nomenclature

$\Re^n$	the n-dimensional real space
I	the unit matrix
A^T	the transport of the matrix A
$\|x\|$	the Euclidean norm of the vector $x \in \Re^n$
$\lambda_{\max}(P)$	the maximum eigenvalue of the matrix P with real eigenvalues
$\sigma(A)$	the spectrum of the matrix A
$P > 0$	the matrix P is a symmetric positive definite matrix

In this paper, we consider the following unified Rossler chaotic system:

$$\dot{x}_1(t) = -x_2(t) - x_3(t), \tag{1a}$$

$$\dot{x}_2(t) = x_1(t) + bx_2(t), \tag{1b}$$

$$\dot{x}_3(t) = -cx_3(t) + a + dx_1(t)x_3(t), \tag{1c}$$

$$y(t) = -(b+1)x_1(t) - (b^2 + b)x_2(t) + x_3(t), \quad \forall\, t \ge 0, \tag{1d}$$

$$[x_1(0) \quad x_2(0) \quad x_3(0)]^T = [x_{10} \quad x_{20} \quad x_{30}]^T, \tag{1e}$$

where $x(t) := [x_1(t) \quad x_2(t) \quad x_3(t)]^T \in \Re^3$ is the state vector, $y(t) \in \Re$ is the system output, $[x_{10} \quad x_{20} \quad x_{30}]^T$ is the initial value, and $a, b, c, d \in \Re$ represent the parameters of the system. The original Rossler chaotic system is a special case of system (1) with $a = b = 0.2, c = 5.7$, and $d = 1$. It is a well-known fact that since states are not always available for direct measurement, particularly in the event of sensor failures, states must be estimated. The objective of this paper is to search a suitable state observer for the system (1) such that the global exponential stability of the resulting error systems can be ensured.

Before presenting the main result, the state reconstructibility is provided as follows.

Definition 1. The system (1) is exponentially state reconstructible if there exist a state observer $f(z, \dot{z}, y) = 0$ and positive numbers κ and α such that

$$\|e(t)\| := \|x(t) - z(t)\| \le \kappa \exp(-\alpha\, t), \quad \forall\, t \ge 0,$$

where $z(t)$ expresses the reconstructed state of the system (1). In this case, the positive number α is called the exponential decay rate.

Now, we are in a position to present the main results for the state observer of system (1).

Theorem 1. The system (1) is exponentially state reconstructible. Furthermore, a suitable state observer is given by

$$\dot{z}_1(t) = -(b+1)z_1(t) - (b^2 + b + 1)z_2(t) - y(t), \tag{2a}$$

$$\dot{z}_2(t) = z_1(t) + bz_2(t), \tag{2b}$$

$$z_3(t) = (b+1)z_1(t) + (b^2 + b)z_2(t) + y(t), \quad \forall\, t \ge 0. \tag{2c}$$

In this case, the guaranteed exponentia decay rate is given by $\alpha := \dfrac{1}{\lambda_{\max}(P)}$, where $P > 0$ is the unique solution to the following Lyapunov equation

$$\begin{bmatrix} -b-1 & -b^2-b-1 \\ 1 & b \end{bmatrix}^T P + P \begin{bmatrix} -b-1 & -b^2-b-1 \\ 1 & b \end{bmatrix} = -2I \, . \tag{3}$$

Proof. For brevity, let us define $A := \begin{bmatrix} -b-1 & -b^2-b-1 \\ 1 & b \end{bmatrix}$ and the observer error

$$e_i(t) := x_i(t) - z_i(t), \forall\, i \in \{1,2,3\} \text{ and } t \ge 0 \, . \tag{4}$$

From (1), (2), and (4), one has

$$\begin{aligned}
\dot{e}_1(t) &= \dot{x}_1(t) - \dot{z}_1(t) \\
&= -x_2(t) - x_3(t) + (b+1)z_1(t) + \left(b^2+b+1\right)z_2(t) + y(t) \\
&= -x_2(t) - \left[y(t) + (b+1)x_1(t) + \left(b^2+b\right)x_2(t)\right] + (b+1)z_1(t) + \left(b^2+b+1\right)z_2(t) + y(t) \\
&= -(b+1)[x_1(t) - x_2(t)] - \left(b^2+b+1\right)[x_2(t) - z_2(t)] \\
&= -(b+1)e_1(t) - \left(b^2+b+1\right)e_2(t), \quad \forall\, t \ge 0,
\end{aligned}$$

$$\begin{aligned}
\dot{e}_2(t) &= \dot{x}_2(t) - \dot{z}_2(t) \\
&= x_1(t) + bx_2(t) - z_1(t) - bz_2(t) \\
&= [x_1(t) - x_2(t)] + b[x_2(t) - z_2(t)] \\
&= e_1(t) + e_2(t), \quad \forall\, t \ge 0.
\end{aligned}$$

It results that

$$\begin{bmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \end{bmatrix} = \begin{bmatrix} -b-1 & -b^2-b-1 \\ 1 & b \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = A \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}, \quad \forall\, t \ge 0. \tag{5}$$

Clearly, one has $\sigma(A) = \left\{ \dfrac{-1+\sqrt{3}i}{2}, \dfrac{-1-\sqrt{3}i}{2} \right\}$. This implies A is Hurwitz and the Lyapunov equation of (3) has the unique posive definite solution *P*. Let

$$V(e_1(t), e_2(t)) = [e_1(t) \quad e_2(t)]P[e_1(t) \quad e_2(t)]^T \, . \tag{6}$$

The time derivative of $V(x(t))$ along the trajectories of the system (5) with (3)-(6) is given by

$$
\begin{aligned}
\frac{dV(e_1(t),e_2(t))}{dt} &= [\dot{e}_1(t) \quad \dot{e}_2(t)]P[e_1(t) \quad e_2(t)]^T + [e_1(t) \quad e_2(t)]P[\dot{e}_1(t) \quad \dot{e}_2(t)]^T \\
&= [-(b+1)e_1(t) \quad -(b^2+b+1)e_2(t)]P[e_1(t) \quad e_2(t)]^T \\
&\quad + [e_1(t) \quad e_2(t)]P[-(b+1)e_1(t) \quad -(b^2+b+1)e_2(t)]^T \\
&= [e_1(t) \quad e_2(t)]A^T P[e_1(t) \quad e_2(t)]^T + [e_1(t) \quad e_2(t)]PA[e_1(t) \quad e_2(t)]^T \\
&= [e_1(t) \quad e_2(t)](A^T P + PA)[e_1(t) \quad e_2(t)]^T \\
&= -2[e_1(t) \quad e_2(t)][e_1(t) \quad e_2(t)]^T \\
&\le \frac{-2}{\lambda_{\max}(P)} V(e_1(t),e_2(t)) \\
&= -2\alpha V(e_1(t),e_2(t)), \quad \forall\, t \ge 0.
\end{aligned}
$$

It follows that

$$
e^{2\alpha t} \cdot \dot{V}(e_1(t),e_2(t)) + 2\alpha e^{2\alpha t} \cdot V(e_1(t),e_2(t)) \le 0
$$

$$
\Rightarrow \quad \frac{d\left[e^{2\alpha t} \cdot V(e_1(t),e_2(t))\right]}{dt} \le 0
$$

$$
\Rightarrow \quad e^{2\alpha t} \cdot V(e_1(t),e_2(t)) - V(e_1(0),e_2(0)) = \int_0^t \frac{d\left[e^{2\alpha t} \cdot V(e_1(t),e_2(t))\right]}{dt} dt \le \int_0^t 0 \; dt = 0
$$

$$
\Rightarrow \quad V(e_1(t),e_2(t)) \le V(e_1(0),e_2(0)) e^{-2\alpha t}, \quad \forall\, t \ge 0. \tag{7}
$$

It can be readily obtained that

$$
\lambda_{\min}(P)\left[e_1^2(t) + e_2^2(t)\right] \le V(e_1(t),e_2(t)) \le V(e_1(0),e_2(0)) e^{-2\alpha t}, \quad \forall\, t \ge 0.
$$

in view of (6) and (7). Hence, it yields

$$
\left[e_1^2(t) + e_2^2(t)\right] \le \frac{V(e_1(0),e_2(0))}{\lambda_{\min}(P)} e^{-2\alpha t}, \quad \forall\, t \ge 0. \tag{8}
$$

Moreover, from (1), (2), and (4), it is easy to see that

$$
\begin{aligned}
e_3(t) &= x_3(t) - z_3(t) \\
&= \left[y(t) + (b+1)x_1(t) + (b^2+b)x_2(t)\right] - \left[(b+1)z_1(t) + (b^2+b)z_2(t) + y(t)\right] \\
&= (b+1)[x_1(t) - z_1(t)] + (b^2+b)[x_2(t) - z_2(t)] \\
&= (b+1)e_1(t) + (b^2+b)e_2(t), \quad \forall\, t \ge 0.
\end{aligned}
$$

It follows that

$$
e_3^2(t) \le 2(b+1)^2 e_1^2(t) + 2(b^2+b)^2 e_2^2(t) \le \delta\left[e_1^2(t) + e_2^2(t)\right], \quad \forall\, t \ge 0, \tag{9}
$$

with $\delta := \max\left\{2(b+1)^2, 2(b^2+b)^2\right\}$. From (8) and (9), we have

$$\|e(t)\| := \sqrt{e_1^2(t)+e_2^2(t)+e_3^2(t)} \le \sqrt{\frac{(1+\delta)V(e_1(0), e_2(0))}{\lambda_{\min}(P)}} e^{-\alpha t}, \quad \forall\, t \ge 0.$$

Consquently, we conclude that the system (2) is a suitable state observer with the guaranted exponential decay rate $\alpha = \dfrac{1}{\lambda_{\max}(P)}$. This completes the proof. □

3. Illustrative example

Consider the unified Rossler chaotic system (1) with $a = b = 0.2$ and $c = 5.7$. From (3) with $b = 0.2$, we have $P = \begin{bmatrix} 2.04 & 1.45 \\ 1.45 & 3.98 \end{bmatrix}$ and $\lambda_{\max}(P) = 4.75$. By Theorem 1, we conclude that the system (1) is exponentially state reconstructible by the state observer

$$\dot{z}_1(t) = -1.2 z_1(t) - 1.24 z_2(t) - y(t), \tag{10a}$$

$$\dot{z}_2(t) = z_1(t) + 0.2 z_2(t), \tag{10b}$$

$$z_3(t) = 1.2 z_1(t) + 0.24 z_2(t) + y(t), \quad \forall\, t \ge 0. \tag{10c}$$

The typical state trajectory of the unified Rossler chaotic system (1) with $a = b = 0.2, c = 5.7$, and $d = 1$ is depicted in Figure 1. In addition, the time response of error states is depicted in Figure 2. From the foregoing simulations results, it is seen that the system (1), with $a = b = 0.2, c = 5.7$, and $d = 1$, is exponentially state reconstructible by the state observer of (10), with the guaranted exponential decay rate $\alpha = \dfrac{1}{\lambda_{\max}(P)} = 0.21$.

4. Conclusions

In this paper, the unified Rossler chaotic system has been addressed and the state observation problem of such a system has been investigated. Based on the time-domain approach with differential and integral inequalities, a suitable state observer for the unified

Rossler chaotic system has been established to assure the global exponential stability of the resulting error system. Besides, the guaranteed exponential decay rate can be accurately estimated. Finally, numerical simulations have been given to demonstrate the feasibility and effectiveness of the obtained results.

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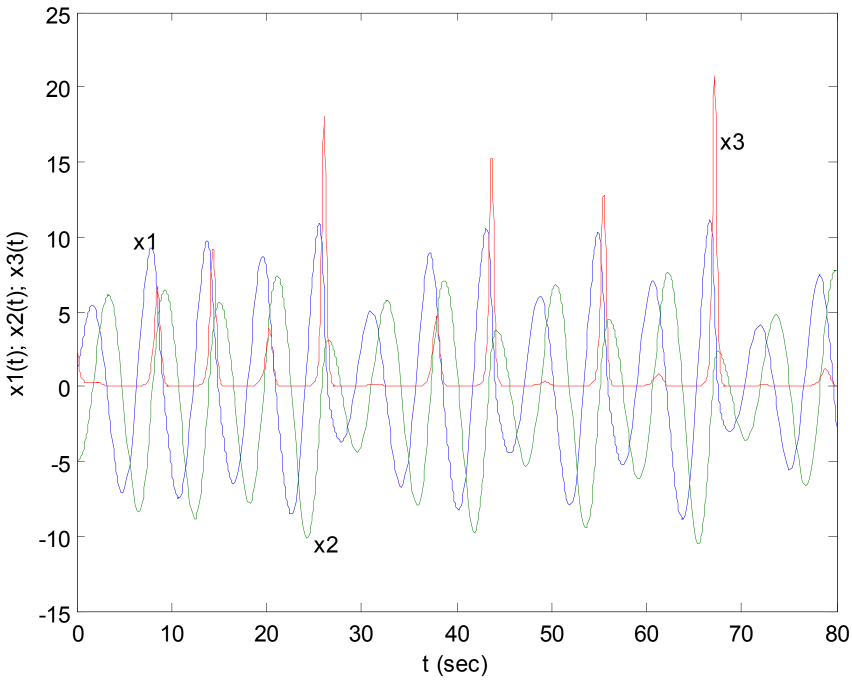

Figure 1: Typical state trajectories of the system (1) with $a = b = 0.2, c = 5.7$ **, and** $d = 1$.

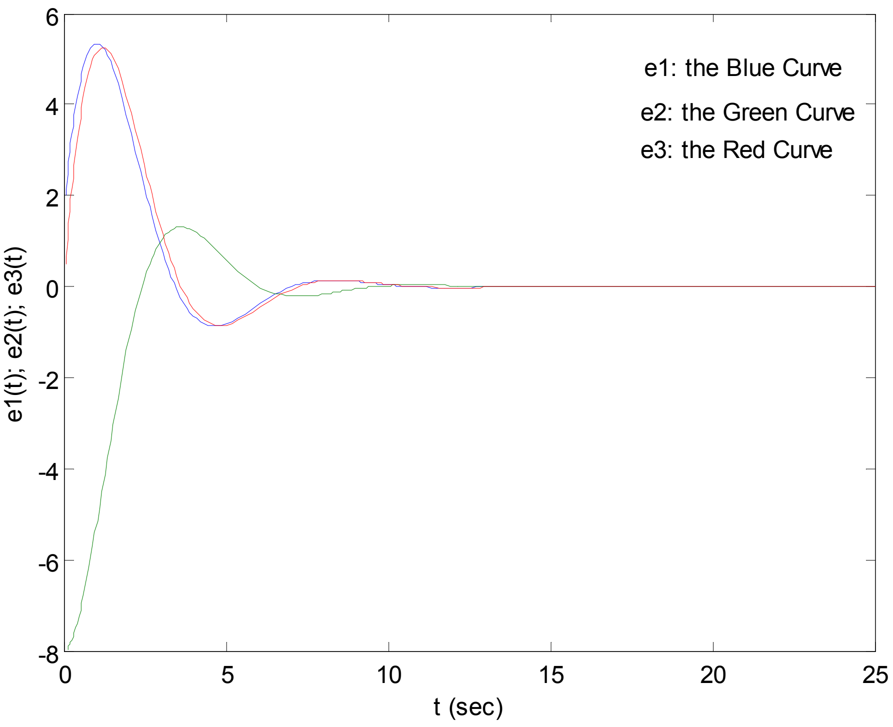

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Figure 2: The time response of error states.

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