SCIREA Journal of Information Science and Systems Science
ISSN: 2995-3936
http://www.scirea.org/journal/ISSS
June 17, 2025
Volume 9, Issue 2, April 2025
https://doi.org/10.54647/isss120405

Knowledge Deduction Based on the Generalized Modal Syllogism A□MM-1

Haowei Shi
Institute of Logic and Information Technology, Sichuan Normal University, Chengdu, China
Email address: 597387580@qq.com

Abstract

This paper first formalizes the non-trivial generalized modal syllogism A□MM-1, then proves its validity according to the truth definitions of the quantifier *all* and *most*. Finally, based on the relevant definitions and facts, the other 27 valid non-trivial generalized modal syllogisms are derived from the syllogism A□MM-1. This knowledge deduction process has logical consistency. It hopes that this study will contribute to the further development of natural language information processing.

Keywords: Generalized Modal Syllogisms, Validity, Knowledge Deduction, Knowledge Mining

1. Introduction

First-order logic only studies the following four quantifiers, *all*, *no*, *not all*, and *some,* which are known as classical quantifiers (Hao, 2024). In fact, there are an infinite number of generalized quantifiers in natural language (Peters and Westerståhl, 2006), such as *most*, *many*, *at least two-thirds*, *both*, etc. A generalized modal syllogism consists of three categorical propositions and non-overlapping modalities (Hao, 2024). A syllogism that includes at least one generalized quantifier and one modality is called a non-trivial generalized modal syllogism. Generalized modal syllogistic reasoning is one of the common forms of reasoning in natural language (Hao and Cao, 2024).

Although scholars have conducted sporadic research on generalized modal syllogisms (Xu and Zhang, 2023a-c; Xu and Wang, 2024), this research needs to be further explored due to the infinite number of generalized quantifiers in natural language. This paper focuses on how to derive other valid generalized modal syllogisms from the syllogism A□MM-1 including the common quantifier *most* in natural language.

2. Preliminaries

In this paper, let *g*, *t*, and *u* be lexical variables, and *D* be their domain. The set composed of *g*, *t*, and *u* is respectively *G, T*, and *U*. Let ϕ, θ, χ, and β be well-formed formulas (abbreviated as wff). '$|G \cap U|$', '□' and '◇' represent respectively the cardinality of the intersection of the set *G* and *U,* a necessary operator, and a possible operator. '$\vdash \chi$' means that the wff χ is provable, '$\chi =_{def} \beta$' that χ can be defined by β. The others are similar. The operators (such as $\neg$, $\rightarrow$, $\wedge$, $\leftrightarrow$) are symbols in set theory (Halmos, 1974).

A quantifier and its three (inner, outer, dual) negative quantifiers form a modern square. For example, Square{*all*}={*all*, *no*, *not all, some*}, and Square{*most*}={*most, fewer than half of the, at most half of the, at least half of the*}. This paper only studies non-trivial generalized modal syllogisms composed of quantifiers from Square{*all*} and Square{*most*}, and modal operators (i.e. □ and/or ◇). Specifically, the syllogisms studied in this paper only involve the following 8 propositions: *all(g, u)*, *no(g, u)*, *not all(g, u)*, *some(g, u), most(g, u), fewer than half of the(g, u), at most half of the(g, u), at least half of the(g, u)*, and they are called Proposition *A*, *E*, *O*, *I*, *M*, *F*, *H*, and *S,* respectively. Therefore, the syllogism A□MM-1 is the abbreviation of the first figure syllogism *all(t, u)*$\wedge$□*most(g, t)*$\rightarrow$*most(g, u).*

Example 1:

Major premise: All persons are mammals.

Minor premise: Most animals that can obtain food through tools are necessarily persons.

Conclusion: Most animals that can obtain food through tools are mammals.

Let t, u, and g be a person, a mammal, and an animal that can obtain food through tools, respectively. Then this syllogism can be formalized as '$all(t, u)\wedge\Box most(g, t)\rightarrow most(g, u)$', which is abbreviated as A□MM-1.

3. A Generalized Modal Syllogism System

Like common logical systems, the generalized modal syllogism system is also composed of primitive symbols, basic axioms, formation rules, deductive rules, and so on.

3.1 Primitive Symbols

(1) lexical variables: g, t, u;

(2) quantifiers: *all, most*;

(3) operators: $\neg$, $\rightarrow$, $\Box$;

(4) brackets: (,).

3.2 Formation Rules

(1) If Q is a quantifier, g and u are lexical variables, then $Q(g, u)$ is a wff;

(2) If θ and ϕ are wffs, then so are $\neg\theta$, $\theta\rightarrow\phi$ and $\Box\theta$;

(3) Only the formulas formed by the above rules are wffs.

3.3 Basic Axioms

A1: If θ is a valid proposition, then $\vdash\theta$;

A2: $\vdash all(t, u)\wedge\Box most(g, t)\rightarrow most(g, u)$ (that is, the syllogism A□MM-1).

3.4 Deductive Rules

Rule 1: From $\vdash(\phi\rightarrow\theta)$ and $\vdash(\theta\wedge\chi\rightarrow\beta)$ infer $\vdash(\phi\wedge\chi\rightarrow\beta)$;

Rule 2: From $\vdash(\theta\wedge\chi\rightarrow\beta)$ and $\vdash(\beta\rightarrow\phi)$ infer $\vdash(\theta\wedge\chi\rightarrow\phi)$;

Rule 3: From $\vdash(\theta\wedge\chi\rightarrow\beta)$ infer $\vdash(\neg\beta\wedge\theta\rightarrow\neg\chi)$.

3.5 Relevant Definitions

D1: $(\theta\wedge\phi)=_{def}\neg(\theta\rightarrow\neg\phi)$;

D2: $(\theta\leftrightarrow\phi) =_{\mathrm{def}} (\theta\rightarrow\phi)\wedge(\phi\rightarrow\theta)$;

D3: $(Q\neg)(g, u) =_{\mathrm{def}} Q(g, D-u)$;

D4: $(\neg Q)(g, u) =_{\mathrm{def}}$ It is not the case that $Q(g, u)$;

D5: $\Diamond Q(g, u) =_{\mathrm{def}} \neg\Box\neg Q(g, u)$;

D6: *all*(g, u) is true iff $G\subseteq U$ is true in any real world;

D7: *some*(g, u) is true iff $G\cap U\neq\varnothing$ is true in any real world;

D8: *no*(g, u) is true iff $G\cap U=\varnothing$ is true in any real world;

D9: *not all*(g, u) is true iff $G\nsubseteq U$ is true in any real world;

D10: *most*(g, u) is true iff $|G\cap U|>0.5|G|$ is true in any real world;

D11: *at most half of the*(g, u) is true iff $|G\cap U|\leq 0.5|G|$ is true in any real world;

D12: *at least half of the*(g, u) is true iff $|G\cap U|\geq 0.5|G|$ is true in any real world;

D13: *fewer than half of the*(g, u) is true iff $|G\cap U|<0.5|G|$ is true in any real world;

D14: $\Box$*most*(g, u) is true iff $|G\cap U|>0.5|G|$ is true in any possible world;

3.6 Relevant Facts

Fact 1 (Inner Negation)

(1.1) *all*$(g, u)\leftrightarrow$*no*$\neg(g, u)$;

(1.2) *no*$(g, u)\leftrightarrow$*all*$\neg(g, u)$;

(1.3) *some*$(g, u)\leftrightarrow$*not all*$\neg(g, u)$;

(1.4) *not all*$(g, u)\leftrightarrow$*some*$\neg(g, u)$;

(1.5) *most*$(g, u)\leftrightarrow$*fewer than half of the*$\neg(g, u)$;

(1.6) *fewer than half of the*$(g, u)\leftrightarrow$*most*$\neg(g, u)$;

(1.7) *at least half of the*$(g, u)\leftrightarrow$*at most half of the*$\neg(g, u)$;

(1.8) *at most half of the*$(g, u)\leftrightarrow$*at least half of the*$\neg(g, u)$.

Fact 2 (Outer Negation)

(2.1) $\neg$*all*$(g, u)\leftrightarrow$*not all*(g, u);

(2.2) $\neg$*not all*(*g*, *u*)$\leftrightarrow$*all*(*g*, *u*);

(2.3) $\neg$*no*(*g*, *u*)$\leftrightarrow$*some*(*g*, *u*);

(2.4) $\neg$*some*(*g*, *u*)$\leftrightarrow$*no*(*g*, *u*);

(2.5) $\neg$*most*(*g*, *u*)$\leftrightarrow$*at most half of the*(*g*, *u*);

(2.6) $\neg$*at most half of the*(*g*, *u*)$\leftrightarrow$*most*(*g*, *u*).

(2.7) $\neg$*fewer than half of the*(*g*, *u*)$\leftrightarrow$*at least half of the*(*g*, *u*);

(2.8) $\neg$*at least half of the*(*g*, *u*)$\leftrightarrow$*fewer than half of the*(*g*, *u*).

Fact 3 (Symmetry):

(3.1) *some*(*g*, *u*)$\leftrightarrow$*some*(*u*, *g*);

(3.2) *no*(*g*, *u*)$\leftrightarrow$*no*(*u*, *g*).

Fact 4 (Subordination):

(4.1) $\vdash$*all(g, u)*$\rightarrow$*some(g, u)*;

(4.2) $\vdash$*no(g, u)*$\rightarrow$*not all(g, u)*;

(4.3) $\vdash$*all(g, u)*$\rightarrow$*most(g, u)*;

(4.4) $\vdash$*most(g, u)*$\rightarrow$*some(g, u)*;

(4.5) $\vdash$*all(g, u)*$\rightarrow$*at least half of the(g, u)*;

(4.6) $\vdash$*at least half of the(g, u)*$\rightarrow$*some(g, u)*;

(4.7) $\vdash$*at least half of the(g, u)*$\rightarrow$*most(g, u)*;

(4.8) $\vdash$*at most half of the(g, u)*$\rightarrow$*fewer than half of the(g, u)*;

(4.9) $\vdash$*fewer than half of the(g, u)*$\rightarrow$*not all(g, u)*;

(4.10) $\vdash$*at most half of the(g, u)*$\rightarrow$*not all(g, u)*;

(4.11) $\vdash$*no(g, u)*$\rightarrow$*fewer than half of the(g, u)*;

(4.12) $\vdash$*no(g, u)*$\rightarrow$*at most half of the(g, u)*;

(4.13) $\vdash\Box Q(g, u)\rightarrow Q(g, u)$;

(4.14) $\vdash\Box Q(g, u)\rightarrow\Diamond Q(g, u)$;

(4.15) $\vdash Q(g, u) \rightarrow \Diamond Q(g, u)$.

Fact 5 (Dual):

(5.1) $\neg\Diamond\neg Q(g, u) \leftrightarrow \Box Q(g, u)$;

(5.2) $\neg\Box\neg Q(g, u) \leftrightarrow \Diamond Q(g, u)$;

(5.3) $\neg\Box Q(g, u) \leftrightarrow \Diamond\neg Q(g, u)$;

(5.4) $\neg\Diamond Q(g, u) \leftrightarrow \Box\neg Q(g, u)$.

The above facts are basic knowledge in generalized quantifier theory (Westerståhl, 2007) and modal logic (Chagrov and Zakharyaschev, 1997), so their proofs are trivial.

4. The Reducibility of Valid Generalized Modal Syllogisms

If one valid syllogism can be deduced from another valid syllogism, it indicates that there is a reducible relationship between these two syllogisms. The following Theorem 1 and Theorem 2 can be proven from the above definitions and facts. Theorem 2 indicates that there are reducible relationships between the valid generalized modal syllogism A□MM-1 and the 27 derived valid syllogisms.

Theorem 1 (A□MM-1): The generalized modal syllogism $all(t, u) \wedge \Box most(g, t) \rightarrow most(g, u)$ is valid.

Proof: Suppose that $all(t, u)$ and $\Box most(g, t)$ are true, it follows that $T \subseteq U$ is true in any real world in line with Definition D6, and $|G \cap T| > 0.5|G|$ is true in any possible world by Definition D14. Because any real world is a possible world, it follows that $|G \cap U| > 0.5|G|$ is true in any real world. Thus, it can be concluded that $most(g, u)$ is true in line with Definition D10, just as desired.

Theorem 2: There are 27 valid generalized modal syllogisms derived from the syllogism A□MM-1:

(2.1) ⊢A□MM-1→A□SM-1

(2.2) ⊢A□MM-1→A□AM-1

(2.3) ⊢A□MM-1→A□MI-1

(2.4) ⊢A□MM-1→A□SM-1→A□SI-1

(2.5) ⊢A□MM-1→□A□MM-1

(2.6) ⊢A□MM-1→A□M◇M-1

(2.7) ⊢A□MM-1→E□MF-1

(2.8) ⊢A□MM-1→E□MF-1→E□MF-2

(2.9) ⊢A□MM-1→E□MF-1→E□MO-1

(2.10) ⊢A□MM-1→E□MF-1→E□MF-2→E□AF-2

(2.11) ⊢A□MM-1→H□MO-3

(2.12) ⊢A□MM-1→H□MO-3→H□SO-3

(2.13) ⊢A□MM-1→H□MO-3→H□AO-3

(2.14) ⊢A□MM-1→AH◇H-2

(2.15) ⊢A□MM-1→AH◇H-2→AE◇H-2

(2.16) ⊢A□MM-1→AH◇H-2→AH◇O-2

(2.17) ⊢A□MM-1→A□SM-1→A□S◇M-1

(2.18) ⊢A□MM-1→A□AM-1→A□A◇M-1

(2.19) ⊢A□MM-1→A□MI-1→A□M◇I-1

(2.20) ⊢A□MM-1→A□SM-1→A□SI-1→A□S◇I-1

(2.21) ⊢A□MM-1→E□MF-1→E□M◇F-1

(2.22) ⊢A□MM-1→E□MF-1→E□MF-2→E□M◇F-2

(2.23) ⊢A□MM-1→E□MF-1→E□MO-1→E□M◇O-1

(2.24) ⊢A□MM-1→E□MF-1→E□MF-2→E□AF-2→E□A◇F-2

(2.25) ⊢A□MM-1→H□MO-3→H□M◇O-3

(2.26) ⊢A□MM-1→H□MO-3→H□SO-3→H□S◇O-3

(2.27) ⊢A□MM-1→H□MO-3→H□AO-3→H□A◇O-3

Proof:

[1] ⊢*all*(*t*, *u*)∧□*most*(*g*, *t*)→*most*(*g*, *u*) (i.e. A□MM-1, basic axiom A2)

[2] ⊢*all*(*t*, *u*)∧□*at least half of the*(*g*, *t*)→*most*(*g*, *u*) (i.e. A□SM-1, by [1], Fact (4.7) and Rule 1)

[3] ⊢*all*(*t*, *u*)∧□*all*(*g*, *t*)→*most*(*g*, *u*) (i.e. A□AM-1, by [1], Fact (4.3) and Rule 1)

[4] ⊢*all*(*t*, *u*)∧□*most*(*g*, *t*)→*some*(*g*, *u*) (i.e. A□MI-1, by [1], Fact (4.4) and Rule 2)

[5] $\vdash all(t, u) \wedge \Box at\ least\ half\ of\ the(g, t) \rightarrow some(g, u)$ (i.e. A□SI-1, by [2], Fact (4.4) and Rule 2)

[6] $\vdash \Box all(t, u) \wedge \Box most(g, t) \rightarrow most(g, u)$ (i.e. □A□MM-1, by [1], Fact (4.13) and Rule 1)

[7] $\vdash all(t, u) \wedge \Box most(g, t) \rightarrow \Diamond most(g, u)$ (i.e. A□M◇M-1, by [1], Fact (4.15) and Rule 2)

[8] $\vdash no\neg(t, u) \wedge \Box most(g, t) \rightarrow fewer\ than\ half\ of\ the\neg(g, u)$ (by [1], Fact(1.1) and (1.5))

[9] $\vdash no(t, D-u) \wedge \Box most(g, t) \rightarrow fewer\ than\ half\ of\ the(g, D-u)$

(i.e. E□MF-1, by [8] and Definition D3)

[10] $\vdash no(D-u, t) \wedge \Box most(g, t) \rightarrow fewer\ than\ half\ of\ the(g, D-u)$

(i.e. E□MF-2, by [9] and Fact (3.2))

[11] $\vdash no(t, D-u) \wedge \Box most(g, t) \rightarrow not\ all(g, D-u)$ (i.e. E□MO-1, by [9], Fact (4.9) and Rule 2)

[12] $\vdash no(D-u, t) \wedge \Box \mathrm{all}(g, t) \rightarrow fewer\ than\ half\ of\ the(g, D-u)$

(i.e. E□AF-2, by [10], Fact (4.3) and Rule 1)

[13] $\vdash \neg most(g, u) \wedge \Box most(g, t) \rightarrow \neg all(t, u)$ (by [1] and Rule 3)

[14] $\vdash at\ most\ half\ of\ the(g, u) \wedge \Box most(g, t) \rightarrow not\ all(t, u)$

(i.e. H□MO-3, by [13], Fact (2.5) and (2.1))

[15] $\vdash at\ most\ half\ of\ the(g, u) \wedge \Box at\ least\ half\ of\ the(g, t) \rightarrow not\ all(t, u)$

(i.e. H□SO-3, by [14], Fact (4.7) and Rule 1)

[16] $\vdash at\ most\ half\ of\ the(g, u) \wedge \Box all(g, t) \rightarrow not\ all(t, u)$

(i.e. H□AO-3, by [14], Fact (4.3) and Rule 1)

[17] $\vdash \neg most(g, u) \wedge all(t, u) \rightarrow \neg \Box most(g, t)$ (by [1] and Rule 3)

[18] $\vdash at\ most\ half\ of\ the(g, u) \wedge all(t, u) \rightarrow \Diamond at\ most\ half\ of\ the(g, t)$

(i.e. AH◇H-2, by [17], Fact (2.5) and (5.3))

[19] $\vdash no(g, u) \wedge all(t, u) \rightarrow \Diamond at\ most\ half\ of\ the(g, t)$

(i.e. AE◇H-2, by [18], Fact (4.12) and Rule 1)

[20] $\vdash at\ most\ half\ of\ the(g, u) \wedge all(t, u) \rightarrow \Diamond not\ all(g, t)$

(i.e. AH◇O-2, by [18], Fact (4.10) and Rule 2)

[21] $\vdash all(t, u) \wedge \Box at\ least\ half\ of\ the(g, t) \rightarrow \Diamond most(g, u)$

(i.e. A□S◇M-1, by [2], Fact (4.15) and Rule 2)

[22] $\vdash all(t, u) \wedge \Box all(g, t) \rightarrow \Diamond most(g, u)$ (i.e. A□A◇M-1, by [3], Fact (4.15) and Rule 2)

[23] $\vdash all(t, u) \wedge \Box most(g, t) \rightarrow \Diamond some(g, u)$ (i.e. A□M◇I-1, by [4], Fact (4.15) and Rule 2)

[24] $\vdash all(t, u) \wedge \Box at\ least\ half\ of\ the(g, t) \rightarrow \Diamond some(g, u)$ (i.e. A□S◇I-1, by [5], Fact (4.15) and Rule 2)

[25] $\vdash \Box all(t, u) \wedge \Box most(g, t) \rightarrow \Diamond most(g, u)$ (i.e. □A□M◇M-1, by [6], Fact (4.15) and Rule 2)

[26] $\vdash no(t, D-u) \wedge \Box most(g, t) \rightarrow \Diamond fewer\ than\ half\ of\ the(g, D-u)$

(i.e. E□M◇F-1, by [9], Fact (4.15) and Rule 2)

[27] $\vdash no(D-u, t) \wedge \Box most(g, t) \rightarrow \Diamond fewer\ than\ half\ of\ the(g, D-u)$

(i.e. E□M◇F-2, by [10], Fact (4.15) and Rule 2)

[28] $\vdash no(t, D-u) \wedge \Box most(g, t) \rightarrow \Diamond not\ all(g, D-u)$ (i.e. E□M◇O-1, by [11], Fact (4.15) and Rule 2)

[29] $\vdash no(D-u, t) \wedge \Box \mathrm{all}(g, t) \rightarrow \Diamond fewer\ than\ half\ of\ the(g, D-u)$

(i.e. E□A◇F-2, by [12], Fact (4.15) and Rule 2)

[30] $\vdash at\ most\ half\ of\ the(g, u) \wedge \Box most(g, t) \rightarrow \Diamond not\ all(t, u)$

(i.e. H□M◇O-3, by [14], Fact (4.15) and Rule 2)

[31] $\vdash at\ most\ half\ of\ the(g, u) \wedge \Box at\ least\ half\ of\ the(g, t) \rightarrow \Diamond not\ all(t, u)$

(i.e. H□S◇O-3, by [15], Fact (4.15) and Rule 2)

[32] $\vdash at\ most\ half\ of\ the(g, u) \wedge \Box all(g, t) \rightarrow \Diamond not\ all(t, u)$

(i.e. H□A◇O-3, by [16], Fact (4.15) and Rule 2)

Theorem 2 states that with the help of relevant deductive rules, definitions, and facts, other valid syllogisms can be mined from a valid generalized modal syllogism. In other words, there is a reducibility between these valid syllogisms. This knowledge mining process belongs to deductive reasoning, therefore, it has logical consistency.

5. Conclusion

This paper firstly formalizes the non-trivial generalized modal syllogism A□MM-1, then proves its validity according to the truth definitions of the quantifier *all* and *most*. Finally, based on the relevant definitions and facts, the other 27 valid non-trivial generalized modal syllogisms are derived from the syllogism A□MM-1. This knowledge deduction process has logical consistency.

Acknowledgement

This work was supported by the National Social Science Fund of China under Grant No.22&ZD295.

References

[1] Hao, Y. J. (2024). The deductibility of categorical syllogisms based on the syllogism EIO-3 from the perspective of dialectics, *SCIREA Journal of Information Science and Systems Science*, 8(5): 194-201.

[2] Peters, S. and Westerståhl, D. (2006). *Quantifiers in Language and Logic.* Oxford: Claredon Press.

[3] Hao, L. H. (2024). The validity of generalized modal syllogisms based on the syllogism E□M◇O-1, *SCIREA Journal of Mathematics*, 9(1): 11-22.

[4] Hao, L. H., and Cao C. Z. (2024). Knowledge reasoning based on the generalized modal syllogism □AM◇M-1 with 'most', *Transactions on Engineering and Computing Sciences*, 12(5): 17-24.

[5] Xu, J., and Zhang X. J. (2023a). How to obtain valid generalized modal syllogisms from valid generalized syllogisms, *Applied Science and Innovative Research*, 7(2): 45-51.

[6] Xu, J., and Zhang X. J. (2023b). The reducibility of generalized modal syllogisms based on □AM◇I-1. *SCIREA Journal of Philosophy*, 3(1): 1-11.

[7] Xu, J., and Zhang X. J. (2023c). How to obtain other valid generalized modal syllogisms from the syllogism □EF◇O-1. *SCIREA Journal of Computer*, 8(2): 63-73.

[8] Xu, J., and Wang, Y. Z. (2024). The validity of generalized modal syllogisms with the quantifiers in Square{no} and Square{most}, *International Journal of Social Science Studies*, 2024, 12(6): 42-47.

[9] Halmos, P. R. (1974). *Naive Set Theory.* New York: Springer-Verlag.

[10] Westerståhl, D. (2007). Quantifiers in formal and natural languages, in: D. M. Gabbay and F. Guenthner(eds.), *Handbook of Philosophical Logic*, Dordrecht: Kluwer Academic Publishers, (14): 223-338.

[11] Chagrov, A., and Zakharyaschev, M. (1997). *Modal Logic*. Oxford: Clarendon Press.