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Resonance Absorption and Transverse Magnetization of an Anti-Ferromagnetic Spin System Interacting with a Phonon Reservoir in the Spin-Wave Region

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Abstract: A form of the transverse magnetic susceptibility is derived and the resonance absorption and transverse magnetization are studied for an anti-ferromagnetic spin system interacting with a phonon reservoir in the spin-wave region, employing the TCLE method of linear response in terms of the non-equilibrium thermo-field dynamics (NETFD), which is reformulated for the revised spin-phonon interaction taken to reflect the energy transfer between the spin system and phonon reservoir. Here, the TCLE method of linear response is a method in which the admittance of a physical system is directly derived from time-convolutionless equations with external driving terms. The approximate formulas of the resonance frequencies, peak-heights (heights of peak) and line half-widths in the resonance region of the power absorption and the amplitude of the expectation value of the transverse magnetization, which is referred as "the magnetization-amplitude", are derived for the anti-ferromagnetic system in a transversely rotating magnetic-field. For an anti-ferromagnetic system of one-dimensional infinite spins in the transversely rotating magnetic-field, the power absorption and magnetization-amplitude are investigated numerically in the region valid for the lowest spin-wave approximation. The approximate formulas of the resonance frequencies, peak-heights and line half-widths, are shown to coincide well with the results investigated calculating numerically the analytic results of the power absorption and magnetization-amplitude in the resonance region, and also are shown to satisfy "the narrowing condition" that as phonon reservoir is damped quickly, the peak-heights increase and the line half-widths decrease, and thus are verified numerically. In the resonance region of the power absorption and magnetization-amplitude, it is shown that as the temperature T becomes high, the resonance frequencies increase slightly, the peak-heights decrease and the line half-widths increase, and that as the wave number k becomes large, the resonance frequencies and peak-heights increase, and the line half-widths decrease, and also that as the spin-magnitude S becomes large, the resonance frequencies and peak-heights of the power absorption and magnetization-amplitude become large. The effects of the memory and initial correlation for the spin system and phonon reservoir, which are represented by the interference terms in the TCLE method and are referred as "the interference effects", are confirmed to increase the power absorption and magnetization-amplitude in the resonance region, and are shown to produce effects that cannot be disregarded for the high temperature, for the non-quickly damped reservoir or for the small wave number k.

Keywords: Anti-Ferromagnetic spin system; Resonance absorption; Transverse magnetic susceptibility; The TCLE method of linear response; Non-equilibrium thermo-field dynamics; Spin-wave method

1 Introduction

The theories of anti-ferromagnetic resonance were macroscopically treated by Nagamiya [1], Kittel and Keffer [2, 3], and were microscopically developed using the spin-wave method [4] by Nakamura [5], Ziman [6], Kubo [7], Akhiezer et al. [8, 9] and Oguchi and Honma [10]. The anti-ferromagnetic resonance was also discussed using the method of the collective motion of spins by Mori and Kawasaki [11], and was studied numerically using the method of calculating the dynamical susceptibility directly by Miyashita et al. [12, 13, 14, 15], and besides its theories were developed by the quantum field theoretical approach of Oshikawa and Affleck to the electron spin resonance in spin-1/2 chains [16, 17, 18]. However, these theories for anti-ferromagnetic resonance do not deal with the effects of the phonon reservoir interacting with the spin systems, and therefore those theories cannot elucidate the damping mechanism of the spin for the case that the spin-spin interactions or the spin-wave interactions are small. In such a case, it is necessary to consider the anti-ferromagnetic spin systems interacting with the phonon reservoirs and to study the effects of the phonon reservoir. Uchiyama et al. [19] proposed a method in which the Kubo formula [20] is calculated using the time-convolution (TC) master equation to study effects of the heat reservoir, and applied it to a two-spin system and a three-spin system. Also, the author and Miyashita [21] formulated the non-equilibrium thermo-field dynamics (NETFD) for an anti-ferromagnetic system of many spins interacting with a phonon reservoir, using the spin-wave method [4, 7]. Recently in Ref. [22], the author derived a form of the transverse magnetic susceptibility and has discussed the resonance absorption for an anti-ferromagnetic system of many spins interacting with a phonon reservoir, using the spin-wave method [4, 7]. It may be an interesting problem to study furthermore the resonance absorption for the anti-ferromagnetic system of many spins interacting with a phonon reservoir.

In the previous paper [22], the author derived a form of the transverse magnetic susceptibility and discussed the resonance absorption for an anti-ferromagnetic system of many spins interacting with a phonon reservoir in the spin-wave region, employing the TCLE method of linear response [23, 24, 25] in terms of the non-equilibrium thermo-field dynamics (NETFD) [26, 27, 28, 29, 30]. Here, the TCLE method of linear response is a method in which the admittance of a physical system is directly derived from time-convolutionless equations with external driving terms [23, 24, 25, 31, 32, 33, 34, 35, 36]. In the previous papers [21, 22], the interaction between the spin system and phonon reservoir was taken to point all of the spins to the "down" direction by the phonon-reservoir field, and thus the spin-phonon interaction does not reflect the energy transfer between the spin system and phonon reservoir at the "down" spin sites. In the problem of collision of the anti-ferromagnetic spin system with the phonon reservoir, it may be necessary to take the spin-phonon interaction to reflect the energy transer between the spin system and phonon reservoir not only at the "up" spin sites but also at the "down" spin sites.

In the present paper, we consider an anti-ferromagnetic spin system with a uniaxial anisotropy energy and an anisotropic exchange interaction under an external static magnetic-field in the spin-wave region, interacting with a phonon reservoir and with an external driving magnetic-field which is a transversely rotating classical field, and study microscopically the power absorption and the transverse magnetization in the resonance region, including the effects of the memory and initial correlation for the spin system and phonon reservoir. We derive a form of the transverse magnetic susceptibility of the anti-ferromagnetic system by employing the TCLE method of linear response [23, 24, 25, 29, 30] in terms of the non-equilibrium thermo-field dynamics (NETFD), which is reformulated for the spin-phonon interaction taken to reflect the energy transfer between the spin system and phonon reservoir not only at the "up" spin sites but also at the "down" spin sites, in the spin-wave approximation employing the spin-wave method of Kubo [7]. We examine analytically the power absorption and the amplitude of the expectation values of the transverse magnetizations, which is referred as "the magnetization-amplitude", in the resonance region of the anti-ferromagnetic system in the spin-wave region, derive the approximate formulas of the resonance frequencies, peak-heights (heights of peak) and line half-widths in the resonance region, and investigate numerically the line shapes for an anti-ferromagnetic system of one-dimensional infinite spins. We also investigate numerically the effects of the memory and initial correlation for the spin system and phonon reservoir, i.e., the interference effects. We use the same symbols and notations as in Refs. [21, 22], and provide the same basic requirements (axioms) as in Refs. [21, 22].

Here, we mention the validity and usefulness of the TCLE method of linear response. In Refs. [34, 35, 36], the relation between the TCLE method and relaxation method for the problem of linear response was analytically examined in the second-order approximation for the interaction between the physical system and heat reservoir, where the relaxation method is the one in which the Kubo formula [20] is calculated for the physical system interacting with the heat reservoir. The admittances derived employing each method were shown to have the same second-order terms and mutually different higher-order terms. The admittances derived employing each method were numerically investigated and were shown to agree well in the resonance region, for a quantum oscillator interacting with a heat reservoir [34] and for a quantum spin interacting with a heat reservoir [35, 37, 38]. This shows that the TCLE method is coincident with the relaxation method in the second-order approximation for the system-reservoir interaction, and that the second-order TCLE method is valid in this approximation. In Refs. [23, 24, 25], the TCLE method and relaxation method were formulated in terms of the NETFD, and the relation between the admittances derived employing each method was analytically examined in the second-order approximation for the interaction between the physical system and heat reservoir [25]. When the relaxation method is employed in the van Hove limit [39] or in the narrowing limit [40], in which the heat reservoir is damped quickly, that is to say, the correlation time τ_c of the heat reservoir is much less than the relaxation time τ_r of the physical system, i.e., $\tau_c \ll \tau_r$, or $\tau_c \to 0$, as done in the formulation of the NETFD [26, 27, 28], the obtained admittance is valid only in that limit and coincides with the one without the interference terms in the admittance derived employing the TCLE method [25, 34, 35]. In the TCLE method, the interference terms are included in the time-convolutionless (TCL) equations with external driving terms [23, 24, 25, 31, 32, 33, 34, 35, 36], represent the effects of the memory and initial correlation for the physical system and heat reservoir, and give the effects of the deviation from the van Hove limit [39] or the narrowing limit [40]. When the TCLE method is employed, the complex admittance of the physical system can be calculated by inserting the interference terms into the results obtained in the van Hove limit [39] or in the narrowing limit [40], in which the NETFD has been formulated [21, 26, 27, 28, 29, 30]. Thus, by employing the NETFD and the TCLE method [23, 24, 25, 29, 30] as done in Refs. [22, 41, 42], the complex admittance of the physical system can be derived including the effects of the memory and initial correlation for the physical system and heat reservoir, i.e., the effects of the motion of the heat reservoir which influence the physical system. As discussed in Ref. [22], one can discuss variations of the peak-heights and line half-widths in the resonance regions of the power-absorption, etc. employing the TCLE method theoretically, because the admittance derived emloying the second-order TCLE method is valid even if the heat reservoir is damped slowly, in the region valid for the second-order perturbation approximation.

In Section 2, we give the Hamiltonian for an anti-ferromagnetic spin system interacting with a phonon reservoir under an external static magnetic-field in the spin-wave region. In Section 3, we derive forms of the transverse magnetic susceptibility and magnetization-amplitude for the anti-ferromagnetic system by employing the TCLE method of linear response in terms of the non-equilibrium thermo-field dynamics (NETFD), which is reformulated for the spin-phonon interaction taken to reflect the energy transfer between the spin system and phonon reservoir in Appendix A, and derive the approximate formulas of the resonance frequencies, peak-heights (heights of peak) and line half-widths in the resonance region of the power absorption and magnetization-amplitude. In Section 4, we investigate numerically the power absorption and magnetization-amplitude in the resonance region of an anti-ferromagnetic system of one-dimensional infinite spins. In Section 5, we give a short summary and some concluding remarks.

2 Model and Hamiltonian of anti-ferromagnetic spin system

We consider an anti-ferromagnetic spin system with a uniaxial anisotropy energy and with an anisotropic exchange interaction under an external static magnetic-field $\vec{H}_z$ in the z direction, in interaction with a phonon reservoir. The anti-ferromagnetic spin system is in the spin-wave region, and we proceed in the spin-wave approximation by employing the spin-wave method of Kubo [7]. We consider a bipartite lattice and denote the sites of sublattices by l and m, where l denote the sites of "up" spins, and m denote the sites of "down" spins. We take the principal axis of the uniaxial anisotropy energy and anisotropic exchange interaction as the z axis, and describe the Hamiltonian $\mathcal{H}_{\mathrm{S}}$ of the anti-ferromagnetic spin system under the external static magnetic-field $\vec{H}_z$ as

$$\mathcal{H}_{\mathrm{S}} = \hbar \sum_{\langle l,\, m\rangle} \left\{ J_1 (S_l^+ S_m^- + S_l^- S_m^+) + 2\, J_2\, S_l^z S_m^z \right\} - \hbar\, \omega_z \Big\{ \sum_l^{N/2} S_l^z + \sum_m^{N/2} S_m^z \Big\} - \hbar K \Big\{ \sum_l^{N/2} (S_l^z)^2 + \sum_m^{N/2} (S_m^z)^2 \Big\}, \quad (2.1)$$

with $S_j^{\pm} = S_j^x \pm i S_j^y$ $(j = l,\, m)$, where ω_z is the Zeeman frequency $\omega_z = \gamma H_z$ with the magnetomechanical ratio γ. In the above Hamiltonian $\mathcal{H}_{\mathrm{S}}$, $\hbar J_1$ and $\hbar J_2$ are the exchange energies, $\hbar K$ is the anisotropy energy, N is the total number of spins, and the summation $\sum_{\langle l,\, m\rangle}$ is taken over all nearest-neighbor pairs. Here, the spin operators $\vec{S}_l$ denote "up" spins of spin magnitude S at sites l, and the spin operators $\vec{S}_m$ denote "down" spins of spin magnitude S at sites m. As done by Kubo [7], we introduce the two kinds of the creation and annihilation operators for the spin deviation. The spin operators $\vec{S}_l$ at up-spin sites l are expressed as

$$S_l^+ = \sqrt{2\,S}\, p_l\, a_l\,, \qquad S_l^- = \sqrt{2\,S}\, a_l^\dagger\, p_l\,, \qquad S_l^z = S - a_l^\dagger\, a_l\,, \quad (2.2)$$

with the Bose operators a_l and $a_l^\dagger$ introduced in Ref. [4], where the operators p_l are defined by

$$p_l = \Big(1 - \frac{a_l^\dagger\, a_l}{2\,S}\Big)^{1/2} = \Big(1 - \frac{n_l}{2\,S}\Big)^{1/2} = 1 - \frac{n_l}{4\,S} - \cdots\,, \qquad (n_l = a_l^\dagger\, a_l). \quad (2.3)$$

The spin operators $\vec{S}_m$ at down-spin sites m are expressed as

$$S_m^+ = \sqrt{2\,S}\, b_m^\dagger\, p_m\,, \qquad S_m^- = \sqrt{2\,S}\, p_m\, b_m\,, \qquad S_m^z = -\,S + b_m^\dagger\, b_m\,, \quad (2.4)$$

with the Bose operators b_m and $b_m^\dagger$ introduced in Ref. [4], where the operators p_m are defined by

$$p_m = \Big(1 - \frac{b_m^\dagger\, b_m}{2\,S}\Big)^{1/2} = \Big(1 - \frac{n_m}{2\,S}\Big)^{1/2} = 1 - \frac{n_m}{4\,S} - \cdots\,, \qquad (n_m = b_m^\dagger\, b_m). \quad (2.5)$$

The Bose operators $a_l^\dagger$ and a_l are the creation and annihilation operators of spin deviation of "up" spins at sites l, respectively, and the Bose operators $b_m^\dagger$ and b_m are the creation and annihilation operators of spin deviation of "down" spins at sites m, respectively. These Bose operators satisfy the commutation relations

$$[\, a_l\,,\ a_{l'}^\dagger\,] = \delta_{l\,l'}\,, \qquad [\, b_m\,,\ b_{m'}^\dagger\,] = \delta_{m\,m'}\,, \quad (2.6)$$

while the other commutators vanish. The Fourier transformations for the Bose operators a_l and b_m are performed as

$$a_l = \sqrt{\frac{2}{N}} \sum_k \bar{a}_k \exp(-\,i\,\vec{k}\cdot\vec{r}_l), \qquad \bar{a}_k = \sqrt{\frac{2}{N}} \sum_l a_l \exp(i\,\vec{k}\cdot\vec{r}_l), \quad (2.7a)$$

$$b_m = \sqrt{\frac{2}{N}} \sum_k \bar{b}_k \exp(i\,\vec{k}\cdot\vec{r}_m), \qquad \bar{b}_k = \sqrt{\frac{2}{N}} \sum_m b_m \exp(-\,i\,\vec{k}\cdot\vec{r}_m), \quad (2.7b)$$

where the transformed operators $\bar{a}_k$ and $\bar{b}_k$ are the Bose operators and satisfy the commutation relations

$$[\, \bar{a}_k\,,\, \bar{a}_{k'}^\dagger\,] = \delta_{kk'}, \qquad [\, \bar{b}_k\,,\, \bar{b}_{k'}^\dagger\,] = \delta_{kk'}, \quad (2.8)$$

while the other commutators vanish. Hereafter, we mainly use the Fourier transformed variables and we omit the overbar "$^-$" unless the meaning is confusing. By substituting (2.2) and (2.4) into Hamiltonian $\mathcal{H}_{\mathrm{S}}$ given by (2.1), by expanding it in accordance with (2.3) and (2.5), and by performing the Fourier transformations (2.7a) and (2.7b), the Hamiltonian $\mathcal{H}_{\mathrm{S}}$ given by (2.1) for the spin system can be divided as $\mathcal{H}_{\mathrm{S}} = \mathcal{H}_{\mathrm{S}0} + \mathcal{H}_{\mathrm{S}1}$ with the free spin-wave Hamiltonian $\mathcal{H}_{\mathrm{S}0}$, which was derived in Ref. [21] in the wave-number representation as

$$\mathcal{H}_{\mathrm{S}0} = 2z\hbar J_1 S \sum_k \left\{ \eta_k (a_k b_k + a_k^\dagger b_k^\dagger) + (\zeta + \kappa + h_z)\, a_k^\dagger a_k + (\zeta + \kappa - h_z)\, b_k^\dagger b_k \right\} - z\hbar J_2 N S^2 - \hbar K N S^2, \quad (2.9)$$

with η_k, ζ, κ and h_z defined by

$$\eta_k = \frac{1}{z}\sum_{\sigma} \exp(i\,\vec{k}\cdot\vec{\sigma}), \qquad \zeta = \frac{J_2}{J_1}\,, \qquad \kappa = \frac{K(2\,S-1)}{2\,zJ_1S}\,, \qquad h_z = \frac{\omega_z}{2\,zJ_1S} = \frac{\gamma\,H_z}{2\,zJ_1S}\,, \tag{2.10}$$

where $\mathcal{H}_{\mathtt{S1}}$ is parts of the higher-order in the spin-wave approximation [21] and represents the interaction among the spin-waves. Here, $\vec{\sigma}$ denotes the vectors to the nearest-neighbour site from each site, and z is the number of the vectors. In order to diagonalize the free spin-wave Hamiltonian $\mathcal{H}_{\mathtt{S0}}$ given by (2.9), the operators a_k, $a_k^\dagger$, b_k, and $b_k^\dagger$ are transformed according to Refs. [7, 10], as

$$a_k = \alpha_k\,\cosh\theta_k - \beta_k^\dagger\,\sinh\theta_k\,, \qquad b_k = -\,\alpha_k^\dagger\,\sinh\theta_k + \beta_k\,\cosh\theta_k\,, \tag{2.11}$$

and their Hermite conjugates, where the operators α_k, $\alpha_k^\dagger$, β_k, and $\beta_k^\dagger$ are the Bose operators and satisfy the commutation relations

$$[\,\alpha_k\,,\,\alpha_{k'}^\dagger\,] = \delta_{kk'}\,, \qquad [\,\beta_k\,,\,\beta_{k'}^\dagger\,] = \delta_{kk'}\,, \tag{2.12}$$

while the other commutators vanish. Taking the choice of θ_k as

$$\sinh 2\theta_k = \eta_k/\sqrt{(\zeta+\kappa)^2-\eta_k^2}\,, \qquad \cosh 2\theta_k = (\zeta+\kappa)/\sqrt{(\zeta+\kappa)^2-\eta_k^2}\,, \tag{2.13}$$

the free spin-wave Hamiltonian $\mathcal{H}_{\mathtt{S0}}$ given by (2.9) takes the diagonal form given by Refs. [21, 22].

We next consider the interaction between the anti-ferromagnetic spin system and phonon reservoir. We assume that each spin interacts only with the reservoir field at the same site as the spin, and thus neglect the spin-phonon interactions among the different sites. We also assume that the phonon reservoir is composed of many phonon which are represented by the Bose operators $R_{l\nu}^a$ and $R_{m\nu}^b$ of mode ν at sites l and m, respectively, and their Hermite conjugates. We perform the Fourier transformations for the phonon operators $R_{l\nu}^a$ and $R_{m\nu}^b$ at the up-spin sites l and down-spin sites m separately, as

$$R_{l\nu}^a = \sqrt{\frac{2}{N}}\sum_{k}\bar{R}_{k\nu}^a\exp(-\,i\,\vec{k}\cdot\vec{r}_l), \qquad \bar{R}_{k\nu}^a = \sqrt{\frac{2}{N}}\sum_{l}R_{l\nu}^a\exp(i\,\vec{k}\cdot\vec{r}_l), \tag{2.14a}$$

$$R_{m\nu}^b = \sqrt{\frac{2}{N}}\sum_{k}\bar{R}_{k\nu}^b\exp(i\,\vec{k}\cdot\vec{r}_m), \qquad \bar{R}_{k\nu}^b = \sqrt{\frac{2}{N}}\sum_{m}R_{m\nu}^b\exp(-\,i\,\vec{k}\cdot\vec{r}_m), \tag{2.14b}$$

and their Hermite conjugates, where the transformed operators $\bar{R}_{k\nu}^a$, $\bar{R}_{k\nu}^b$ and their Hermite conjugates are the Bose operators and satisfy the commutation relations

$$[\,\bar{R}_{k\nu}^a\,,\,\bar{R}_{k'\nu'}^{a\dagger}\,] = \delta_{kk'}\delta_{\nu\nu'}\,, \qquad [\,\bar{R}_{k\nu}^b\,,\,\bar{R}_{k'\nu'}^{b\dagger}\,] = \delta_{kk'}\delta_{\nu\nu'}\,, \tag{2.15}$$

while the other commutators vanish. Hereafter, we mainly use the Fourier transformed variables and we omit "$^-$" unless the meaning is confusing. The interaction Hamiltonian $\mathcal{H}_{\mathtt{SR}}$ between the spin system and phonon reservoir is taken as

$$\begin{aligned}\mathcal{H}_{\mathtt{SR}} = &-\frac{\hbar}{2}\Big\{\sum_{l,\,\nu}\big(g_{1\nu}^*\,S_l^+\,R_{l\nu}^{a\dagger} + g_{1\nu}\,S_l^-\,R_{l\nu}^a\big) + \sum_{m,\,\nu}\big(g_{1\nu}\,S_m^+\,R_{m\nu}^b + g_{1\nu}^*\,S_m^-\,R_{m\nu}^{b\dagger}\big)\Big\}\\ &-\hbar\,\Big\{\sum_{l,\,\nu}g_{2\nu}\,S_l^z\,R_{l\nu}^{a\dagger}\,R_{l\nu}^a + \sum_{m,\,\nu}g_{2\nu}\,S_m^z\,R_{m\nu}^{b\dagger}\,R_{m\nu}^b\Big\},\end{aligned} \tag{2.16a}$$

$$\begin{aligned}= &-\frac{\hbar}{2}\sum_{k,\,\nu}\big\{\sqrt{2\,S}\,(g_{1\nu}^*\,a_kR_{k\nu}^{a\dagger} + g_{1\nu}\,a_k^\dagger R_{k\nu}^a) + \sqrt{2\,S}\,(g_{1\nu}\,b_k^\dagger R_{k\nu}^b + g_{1\nu}^*\,b_kR_{k\nu}^{b\dagger})\big\} + \cdots\\ &-\hbar\sum_{k,\,\nu}g_{2\nu}\Big(S - \frac{2}{N}\sum_{k'}a_{k'}^\dagger a_{k'}\Big)R_{k\nu}^{a\dagger}R_{k\nu}^a - \hbar\sum_{k,\,\nu}g_{2\nu}\Big(\frac{2}{N}\sum_{k'}b_{k'}^\dagger b_{k'} - S\Big)R_{k\nu}^{b\dagger}R_{k\nu}^b + \cdots\,,\end{aligned} \tag{2.16b}$$

where $g_{1\nu}$ and $g_{2\nu}$ are the coupling constants between the spin and the phonon of mode ν. In the derivation of (2.16b), we have substituted (2.2) and (2.4) into (2.16a) and have expanded it according to (2.3) and (2.5). In (2.16b), the first "$\cdots$" denotes the higher-order parts of the first term of (2.16a) in the spin-wave approximation, and the second "$\cdots$" denotes the off-diagonal parts in the Fourier transformation of the second term of (2.16a). The above spin-phonon interaction Hamiltonian $\mathcal{H}_{\mathtt{SR}}$ reflects the energy transfer between the spin system and phonon reservoir, and is different from the spin-phonon interaction taken in Refs. [21, 22], which does not reflect the energy transfer between the spin system and phonon reservoir at the sites m of "down" spins, because the spin-phonon interaction taken in Refs. [21, 22] was taken to point all of the spins to the "down" direction by the phonon-reservoir field.

In the spin-phonon interaction $\mathcal{H}_{\mathrm{SR}}$ given by (2.16), we assume that same as the x and y components of the spin, the z component of the spin is coupled only with the phonon operators of the same wave-number as the spin. We also assume that the thermal equilibrium value of the phonon number of the wave number k at the up-spin sites l coincides with that of the wave number k at the down-spin sites m in the phonon reservoir, and put

$$\sum_{\nu} g_{2\nu} \langle 1_{\mathrm{R}} | R_{k\nu}^{a\dagger} R_{k\nu}^{a} | \rho_{\mathrm{R}} \rangle = \sum_{\nu} g_{2\nu} \langle 1_{\mathrm{R}} | R_{k\nu}^{b\dagger} R_{k\nu}^{b} | \rho_{\mathrm{R}} \rangle = \sum_{\nu} g_{2\nu} \langle 1_{\mathrm{R}} | R_{k\nu}^{\dagger} R_{k\nu} | \rho_{\mathrm{R}} \rangle, \tag{2.17}$$

with the Bose operators $R_{k\nu}$ and $R_{k\nu}^{\dagger}$, where $\langle 1_{\mathrm{R}} | \cdots | \rho_{\mathrm{R}} \rangle = \mathrm{tr}_{\mathrm{R}} \cdots \rho_{\mathrm{R}}$ is the notaion of thermo-field dynamics. Here, ρ_{R} is the normalized, time-independent density operator for the phonon reservoir with the Hamiltonian $\mathcal{H}_{\mathrm{R}}$, and is given by

$$\rho_{\mathrm{R}} = \exp(-\beta \mathcal{H}_{\mathrm{R}}) / \langle 1_{\mathrm{R}} | \exp(-\beta \mathcal{H}_{\mathrm{R}}) | 1_{\mathrm{R}} \rangle = \exp(-\beta \mathcal{H}_{\mathrm{R}}) / \, \mathrm{tr}_{\mathrm{R}} \exp(-\beta \mathcal{H}_{\mathrm{R}}), \tag{2.18}$$

which is the thermal equilibrium density operator at temperature $T = (k_{\mathrm{B}} \beta)^{-1}$, where notation tr_{R} denotes the trace operation in the space of the phonon reservoir. We do not specify the Hamiltonian $\mathcal{H}_{\mathrm{R}}$ of the phonon reservoir explicitly. For the later convenience, we renormalize the free spin-wave Hamiltonian $\mathcal{H}_{\mathrm{S0}}$, the free spin-wave energies $\hbar \epsilon_k^{\pm}$ and the spin-phonon interaction $\mathcal{H}_{\mathrm{SR}}$, as follows

$$\mathcal{H}_{\mathrm{S0}} = \hbar \sum_{k} \Big\{ \epsilon_k^{+} \, \alpha_k^{\dagger} \, \alpha_k + \epsilon_k^{-} \, \beta_k^{\dagger} \, \beta_k + \frac{1}{2} \, (\epsilon_k^{+} + \epsilon_k^{-}) \Big\} - z \, \hbar \, J_1 \, N \, S \, (\zeta + \kappa) - z \, \hbar \, J_2 \, N \, S^2 - \hbar \, K \, N \, S^2, \tag{2.19}$$

$$\hbar \, \epsilon_k^{\pm} = 2 \, z \, \hbar \, J_1 \, S \, \big\{ \sqrt{(\zeta + \kappa)^2 - \eta_k^2} \pm h_z \big\} \pm \hbar \sum_{\nu} g_{2\nu} \, \langle 1_{\mathrm{R}} | \, R_{k\nu}^{\dagger} R_{k\nu} \, | \rho_{\mathrm{R}} \rangle, \tag{2.20}$$

$$\begin{aligned} \mathcal{H}_{\mathrm{SR}} = & - \hbar \, \sqrt{S/2} \sum_{k,\, \nu} \big\{ g_{1\nu}^{*} \, (a_k R_{k\nu}^{a\dagger} + b_k R_{k\nu}^{b\dagger}) + g_{1\nu} \, (a_k^{\dagger} R_{k\nu}^{a} + b_k^{\dagger} R_{k\nu}^{b}) \big\} \\ & - \hbar \sum_{k,\, \nu} g_{2\nu} \big\{ (S - a_k^{\dagger} a_k)(R_{k\nu}^{a\dagger} R_{k\nu}^{a} - \langle 1_{\mathrm{R}} | R_{k\nu}^{a\dagger} R_{k\nu}^{a} | \rho_{\mathrm{R}} \rangle) + (b_k^{\dagger} b_k - S)(R_{k\nu}^{b\dagger} R_{k\nu}^{b} - \langle 1_{\mathrm{R}} | R_{k\nu}^{b\dagger} R_{k\nu}^{b} | \rho_{\mathrm{R}} \rangle) \big\}, \end{aligned} \tag{2.21}$$

where we have ignored the higher-order parts in the spin-wave approximation, the off-diagonal parts and the wave-number mixing in $\mathcal{H}_{\mathrm{SR}}$. Hereafter, we use $\mathcal{H}_{\mathrm{S0}}$, $\hbar \epsilon_k^{\pm}$ and $\mathcal{H}_{\mathrm{SR}}$ given by (2.19) – (2.21), respectively, for the free spin-wave Hamiltonian, the free spin-wave energies and the spin-phonon interaction. We besides assume that the thermal equilibrium values of the phonon operators vanish, i.e., $\langle 1_{\mathrm{R}} | R_{k\nu}^{a(b)} | \rho_{\mathrm{R}} \rangle = \langle 1_{\mathrm{R}} | R_{k\nu}^{a(b)\dagger} | \rho_{\mathrm{R}} \rangle = 0$. Then, we have

$$\langle 1_{\mathrm{R}} | \mathcal{H}_{\mathrm{SR}} | \rho_{\mathrm{R}} \rangle = 0, \qquad \langle 1_{\mathrm{R}} | \hat{\mathcal{H}}_{\mathrm{SR}} | \rho_{\mathrm{R}} \rangle = 0, \qquad [\hat{\mathcal{H}}_{\mathrm{SR}} = (\mathcal{H}_{\mathrm{SR}} - \tilde{\mathcal{H}}_{\mathrm{SR}}^{\dagger}) / \hbar], \tag{2.22}$$

where $\hat{\mathcal{H}}_{\mathrm{SR}}$ are the renormalized hat-Hamiltonian defined by $\hat{\mathcal{H}}_{\mathrm{SR}} = (\mathcal{H}_{\mathrm{SR}} - \tilde{\mathcal{H}}_{\mathrm{SR}}^{\dagger}) / \hbar$ [25]. The renormalized free spin-wave energies $\hbar \epsilon_k^{\pm}$ given by (2.20) include the thermal equilibrium values of the phonon number, which depend on temperature T in general. We assume that the phonon operators for each wave number and each mode are mutually independent and assume that

$$\langle 1_{\mathrm{R}} | R_{k\nu}^{a}(t) R_{k\nu}^{a} | \rho_{\mathrm{R}} \rangle = \langle 1_{\mathrm{R}} | R_{k\nu}^{a\dagger}(t) R_{k\nu}^{a\dagger} | \rho_{\mathrm{R}} \rangle = \langle 1_{\mathrm{R}} | R_{k\nu}^{b}(t) R_{k\nu}^{b} | \rho_{\mathrm{R}} \rangle = \langle 1_{\mathrm{R}} | R_{k\nu}^{b\dagger}(t) R_{k\nu}^{b\dagger} | \rho_{\mathrm{R}} \rangle = 0, \tag{2.23a}$$

$$\langle 1_{\mathrm{R}} | \tilde{R}_{k\nu}^{a}(t) \tilde{R}_{k\nu}^{a} | \rho_{\mathrm{R}} \rangle = \langle 1_{\mathrm{R}} | \tilde{R}_{k\nu}^{a\dagger}(t) \tilde{R}_{k\nu}^{a\dagger} | \rho_{\mathrm{R}} \rangle = \langle 1_{\mathrm{R}} | \tilde{R}_{k\nu}^{b}(t) \tilde{R}_{k\nu}^{b} | \rho_{\mathrm{R}} \rangle = \langle 1_{\mathrm{R}} | \tilde{R}_{k\nu}^{b\dagger}(t) \tilde{R}_{k\nu}^{b\dagger} | \rho_{\mathrm{R}} \rangle = 0, \tag{2.23b}$$

with the Heisenberg operators $R_{k\nu}^{a(b)}(t) = \exp(i \hat{\mathcal{H}}_{\mathrm{R}} t) R_{k\nu}^{a(b)} \exp(-i \hat{\mathcal{H}}_{\mathrm{R}} t)$, $\tilde{R}_{k\nu}^{a(b)}(t) = \exp(i \hat{\mathcal{H}}_{\mathrm{R}} t) \tilde{R}_{k\nu}^{a(b)} \exp(-i \hat{\mathcal{H}}_{\mathrm{R}} t)$, and their Hermite conjugates, which are the Heisenberg operators in the space of the phonon reservoir. We also assume that the phonon operators at the up-spin sites l are independent of the phonon operators at the down-spin sites m, e.g., $\langle 1_{\mathrm{R}} | R_{k\nu}^{a}(t) R_{k\nu}^{b} | \rho_{\mathrm{R}} \rangle = \langle 1_{\mathrm{R}} | R_{k\nu}^{a\dagger}(t) R_{k\nu}^{b} | \rho_{\mathrm{R}} \rangle = 0$. We besides assume that the correlation function for the phonon operator with the wave number k at the up-spin sites l coincides with the correlation function for the phonon operator with the wave number k at the down-spin sites m, and put

$$\sum_{\nu} |g_{1\nu}|^2 \langle 1_{\mathrm{R}} | R_{k\nu}^{a\dagger}(t) R_{k\nu}^{a} | \rho_{\mathrm{R}} \rangle = \sum_{\nu} |g_{1\nu}|^2 \langle 1_{\mathrm{R}} | R_{k\nu}^{b\dagger}(t) R_{k\nu}^{b} | \rho_{\mathrm{R}} \rangle = \sum_{\nu} |g_{1\nu}|^2 \langle 1_{\mathrm{R}} | R_{k\nu}^{\dagger}(t) R_{k\nu} | \rho_{\mathrm{R}} \rangle, \tag{2.24a}$$

$$\sum_{\nu} |g_{1\nu}|^2 \langle 1_{\mathrm{R}} | R_{k\nu}^{a}(t) R_{k\nu}^{a\dagger} | \rho_{\mathrm{R}} \rangle = \sum_{\nu} |g_{1\nu}|^2 \langle 1_{\mathrm{R}} | R_{k\nu}^{b}(t) R_{k\nu}^{b\dagger} | \rho_{\mathrm{R}} \rangle = \sum_{\nu} |g_{1\nu}|^2 \langle 1_{\mathrm{R}} | R_{k\nu}(t) R_{k\nu}^{\dagger} | \rho_{\mathrm{R}} \rangle, \tag{2.24b}$$

$$\begin{aligned} & \sum_{\nu} g_{2\nu}^2 \, \langle 1_{\mathrm{R}} | \Delta(R_{k\nu}^{a\dagger}(t) R_{k\nu}^{a}(t)) \Delta(R_{k\nu}^{a\dagger} R_{k\nu}^{a}) | \rho_{\mathrm{R}} \rangle = \sum_{\nu} g_{2\nu}^2 \, \langle 1_{\mathrm{R}} | \Delta(R_{k\nu}^{b\dagger}(t) R_{k\nu}^{b}(t)) \Delta(R_{k\nu}^{b\dagger} R_{k\nu}^{b}) | \rho_{\mathrm{R}} \rangle \\ & = \sum_{\nu} g_{2\nu}^2 \, \langle 1_{\mathrm{R}} | \Delta(R_{k\nu}^{\dagger}(t) R_{k\nu}(t)) \Delta(R_{k\nu}^{\dagger} R_{k\nu}) | \rho_{\mathrm{R}} \rangle, \end{aligned} \tag{2.24c}$$

where we have put, for example, as $\Delta(R_{k\nu}^{\dagger}(t) R_{k\nu}(t)) = R_{k\nu}^{\dagger}(t) R_{k\nu}(t) - \langle 1_{\mathrm{R}} | R_{k\nu}^{\dagger} R_{k\nu} | \rho_{\mathrm{R}} \rangle$ and $\Delta(R_{k\nu}^{\dagger} R_{k\nu}) = R_{k\nu}^{\dagger} R_{k\nu} - \langle 1_{\mathrm{R}} | R_{k\nu}^{\dagger} R_{k\nu} | \rho_{\mathrm{R}} \rangle$. As done in Refs. [21, 22], we assume that the phonon correlation function given by (2.24c) is real.

In Appendix A, we reformulate the non-equilibrium thermo-field dynamics (NETFD) for the spin-phonon interaction (2.21) taken to reflect the energy transfer between the spin system and phonon reservoir.

In the last of this section, we check the ground state of the anti-ferromagnetic spin system. In the lowest spin-wave approximation, the Hamiltonian $\mathcal{H}_{\mathtt{S}0}$ of the spin system, which is given by (2.19) and (2.20), can be rewritten as

$$\mathcal{H}_{\mathtt{S}0} = 2z\hbar J_1 S\,(\zeta+\kappa)\sum_k \big\{\sqrt{1-\tanh^2(2\theta_k)}-1\big\} + \sum_k\{\epsilon_k^+\,\alpha_k^\dagger\alpha_k + \epsilon_k^-\,\beta_k^\dagger\beta_k\} - z\hbar J_2 NS^2 - \hbar KNS^2, \tag{2.25}$$

which we have put $\tanh(2\theta_k)=\eta_k/(\zeta+\kappa)$ according to (2.13). Then, the ground state energy $E_{\mathtt{S}0}^{\mathtt{G}}$ of the spin system in the lowest spin-wave approximation is given by

$$E_{\mathtt{S}0}^{\mathtt{G}} = -z\hbar J_2 NS^2 - \hbar KNS^2 + 2z\hbar J_1 S\,(\zeta+\kappa)\sum_k \big\{\sqrt{1-\tanh^2(2\theta_k)}-1\big\}, \tag{2.26}$$

which is smaller than the energy $-z\hbar J_2 NS^2 - \hbar KNS^2$ of the Neel ordered state with the anisotropy energy $\hbar K$, because the third term of $E_{\mathtt{S}0}^{\mathtt{G}}$ given by (2.26) is negative according to $\{\sqrt{1-\tanh^2(2\theta_k)}-1\}<0$. Thus, the ground state of the spin system in the lowest spin-wave approximation is lower than the Neel ordered state with the anisotropy energy [43]. In the case of an anti-ferromagnetic spin system with the isotropic exchange interaction and without anisotropy energy, i.e., $\zeta=1,\ K=0,\ \kappa=0$, the ground state energy of the spin system in the lowest spin-wave approximation becomes

$$E_{\mathtt{S}0}^{\mathtt{G}} = -z\hbar JNS^2 + 2z\hbar JS\sum_k \big\{\sqrt{1-\eta_k^2}-1\big\}, \qquad (J=J_1=J_2\,;\ K=0), \tag{2.27}$$

which is smaller than the energy $-z\hbar JNS^2$ of the Neel ordered state [43], where we have put $J=J_1=J_2$, and thus the ground state of the spin system is lower than the Neel ordered state.

3 Resonance absorption and transverse magnetization

In this section, we derive forms of the transverse magnetic susceptibility, the expectation value of the transverse magnetization and its amplitude for the anti-ferromagnetic spin system interacting with the phonon reservoir, by employing the TCLE method of linear response in terms of the non-equilibrium thermo-field dynamics (NETFD) reformulated in Appendix A. The TCLE method of linear response was formulated in terms of the NETFD in Refs. [23, 24, 25], and it was surveyed in Appendix A of Ref. [22]. We consider the case that the external driving magnetic-field $\vec{H}_j(t)$ at site j is a transversely rotating classical field:

$$\vec{H}_j(t) = (H_j\cos\omega t\,,\ -H_j\sin\omega t\,,\ 0), \qquad (H_j^* = H_j\,;\ j=l,m), \tag{3.1}$$

and take the interaction $\mathcal{H}_{\mathrm{ed}}(t)$ of the spin system with the external driving field as

$$\begin{aligned}\mathcal{H}_{\mathrm{ed}}(t) &= -\,\hbar\,\gamma\sum_j \vec{S}_j\cdot\vec{H}_j(t) = -\,(1/2)\,\hbar\,\gamma\sum_j\{S_j^+\,H_j^-(t) + S_j^-\,H_j^+(t)\}, \qquad (j=l,m),\\ &= -\,\frac{\hbar\,\gamma}{2}\Big\{\sum_l H_l\,\{S_l^+\exp(i\,\omega\,t) + S_l^-\exp(-\,i\,\omega\,t)\} + \sum_m H_m\{S_m^+\exp(i\,\omega\,t) + S_m^-\exp(-\,i\,\omega\,t)\}\Big\},\\ &= -\,\hbar\,\gamma\,\sqrt{S/2}\,\Big\{\sum_l H_l\,\{a_l\exp(i\,\omega\,t) + a_l^\dagger\exp(-\,i\,\omega\,t)\} + \sum_m H_m\{b_m^\dagger\exp(i\,\omega\,t) + b_m\exp(-\,i\,\omega\,t)\}\Big\}\\ &\quad + \cdots,\end{aligned} \tag{3.2}$$

with $H_j^\pm(t) = H_j^x(t) \pm i\,H_j^y(t) = H_j\exp(\mp\,i\,\omega\,t)$, where we have performed the transformations (2.2) and (2.4) and the expansions (2.3) and (2.5). Here, "$\cdots$" denotes the higher-order parts in the spin-wave approximation, and we neglect the higher-order parts in the following. By performing the Fourier transformations (2.7a) and (2.7b), the above interaction $\mathcal{H}_{\mathrm{ed}}(t)$ can be rewritten in the wave-number representation as

$$\mathcal{H}_{\mathrm{ed}}(t) = -\,\hbar\,\gamma\,\sqrt{S/2}\sum_k\big\{(a_k + b_k^\dagger)\,\bar{H}_k\exp(i\,\omega\,t) + (a_k^\dagger + b_k)\,\bar{H}_k^*\exp(-\,i\,\omega\,t)\big\}, \tag{3.3}$$

where $\bar{H}_k$ is the Fourier transformation of H_j $[=H_j^*]$:

$$H_j = \sqrt{2/N}\sum_k \bar{H}_k\exp(i\,\vec{k}\cdot\vec{r}_j), \qquad \bar{H}_k = \sqrt{2/N}\sum_j H_j\exp(-\,i\,\vec{k}\cdot\vec{r}_j), \qquad (j=l,m). \tag{3.4}$$

Hereafter, we mainly use the Fourier transformed variables and we omit "$^{-}$" unless the meaning is confusing. When the external driving magnetic-field $\vec{H}_j(t)$ is uniform in space, i.e., $H_j = H$, we have $H_k = H_0\,\delta_{k0}$ and $H_0 = H_0^* = \sqrt{N/2}\,H$, and the form of the interaction $\mathcal{H}_{\rm ed}(t)$ becomes

$$\mathcal{H}_{\rm ed}(t) = -\,(1/2)\,\hbar\,\gamma\,H\sqrt{N\,S}\,\big\{(a_0 + b_0^\dagger)\exp(i\,\omega\,t) + (a_0^\dagger + b_0)\exp(-\,i\,\omega\,t)\big\}. \tag{3.5}$$

The transverse magnetic susceptibility $\chi_{S_k^+ S_k^-}(\omega)$ for the anti-ferromagnetic spin system specified in Section 2, is given by employing the TCLE method formulated in terms of the NETFD [22, 23, 24, 25], as

$$\chi_{S_k^+ S_k^-}(\omega) = \frac{1}{2}\int_0^\infty dt\,\langle 1_{\tt S}|\,\gamma\,\hbar\,S_k^+\,U(t)\exp_{\leftarrow}\Big\{-\,i\int_0^t d\tau\,\hat{\mathcal{H}}_{\tt S1}(\tau)\Big\}$$
$$\times\,\big\{i\,(\gamma/2)(S_k^- - \tilde{S}_k^+)|\rho_0\rangle + |D^{(2)}_{S_k^-}[\omega]\rangle\big\}\exp(i\,\omega\,t), \tag{3.6}$$

in the the second-order approximation for the spin-phonon interaction, where $U(t)$ and $\hat{\mathcal{H}}_{\tt S1}(t)$ are defined by (A.21) and (A.22), respectively, and $|\rho_0\rangle$ is defined by $|\rho_0\rangle = \langle 1_{\tt R}|\rho_{\tt TE}\rangle$ for $\rho_{\tt TE}$ given by (A.3). Here, $S_k^\pm$ are the Fourier transformations of the sin operators $S_j^\pm$, i.e.,

$$S_j^\pm = \sqrt{2/N}\sum_k \bar{S}_k^\pm\exp(\mp\,i\,\vec{k}\cdot\vec{r}_j), \qquad \bar{S}_k^\pm = \sqrt{2/N}\sum_j S_j^\pm\exp(\pm\,i\,\vec{k}\cdot\vec{r}_j), \qquad (j = l, m), \tag{3.7}$$

with $\bar{S}_k^\pm \Rightarrow S_k^\pm$, i.e., "$^{-}$" is omitted hereafter unless the meaning is confusing. The above transverse susceptibility $\chi_{S_k^+ S_k^-}(\omega)$ is valid even if the spin system is interacting with a non-quickly damped phonon-reservoir. Here, the interference thermal state $|D^{(2)}_{S_k^-}[\omega]\rangle$ represents the effects of the memory and initial correlation for the spin system and phonon reservoir, and can be written as [22, 23, 24, 25]

$$|D^{(2)}_{S_k^-}[\omega]\rangle = i\,\gamma\,\sqrt{S/2}\int_0^\infty d\tau\int_0^\tau ds\,\big\{\langle 1_{\tt R}|\hat{\mathcal{H}}_{\tt SR}\exp\{-\,i\,\hat{\mathcal{H}}_0\,\tau\}\hat{\mathcal{H}}_{\tt SR}\exp\{i\,\hat{\mathcal{H}}_0(\tau - s)\}$$
$$\times\,\big(a_k^\dagger - \tilde{a}_k + b_k - \tilde{b}_k^\dagger\big)|\rho_0\rangle|\rho_{\tt R}\rangle\exp(i\,\omega\,s)$$
$$-\,\langle 1_{\tt R}|\hat{\mathcal{H}}_{\tt SR}\exp\{-\,i\,\hat{\mathcal{H}}_0\,s\}\big(a_k^\dagger - \tilde{a}_k + b_k - \tilde{b}_k^\dagger$$
$$\times\,\exp\{i\,\hat{\mathcal{H}}_0\cdot(s - \tau)\}\hat{\mathcal{H}}_{\tt SR}|\rho_0\rangle|\rho_{\tt R}\rangle\exp(i\,\omega\,s)\big\}, \tag{3.8}$$

with $\mathcal{H}_0 = \mathcal{H}_{\tt S0} + \mathcal{H}_{\tt R}$, where we have neglected the higher-order parts in the spin-wave approximation. The above interference thermal state $|D^{(2)}_{S_k^-}[\omega]\rangle$ is calculated by substituting (2.21) into (3.8) in Appendix B, can be expressed as (B.2) by using the correlation functions $\phi_k^{+-}(\epsilon)$, $\phi_k^{-+}(\epsilon)$ and $\phi_k^{zz}(\epsilon)$ defined by (A.25a) – (A.25c), and can be rewritten as

$$|D^{(2)}_{S_k^-}[\omega]\rangle = \gamma\,\sqrt{S/2}\,(\cosh\theta_k - \sinh\theta_k)\{|D^{(2)}_{k1}[\omega]\rangle/(2\,(\omega - \epsilon_k^+)) + |D^{(2)}_{k2}[\omega]\rangle/(2\,(\omega + \epsilon_k^-))\}, \tag{3.9}$$

with $|D^{(2)}_{k1}[\omega]\rangle$ and $|D^{(2)}_{k2}[\omega]\rangle$ defined by

$$\begin{aligned}
|D^{(2)}_{k1}[\omega]\rangle = {} & S\,(\cosh 2\theta_k + 1)\,(\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle\,\{\Phi_k^+(\omega) - \Phi_k^+(\epsilon_k^+)\} \\
& + S\,(\cosh 2\theta_k - 1)\,(\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle\,\{\Phi_k^-(\omega) - \Phi_k^-(\epsilon_k^+)\} \\
& - S\sinh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle\,\{(\Phi_k^+(\omega) - \Phi_k^+(\epsilon_k^+)) + (\Phi_k^-(\omega) - \Phi_k^-(\epsilon_k^+))\} \\
& + \sinh 2\theta_k\cosh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle\,\{\Psi_k(\omega + \epsilon_k^-) - \Psi_k\} \\
& + (\cosh^2 2\theta_k + 1)\,(\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle\,\{\Psi_k(\omega - \epsilon_k^+) - \Psi_k^0\} \\
& + (\alpha_k\beta_k + \alpha_k^\dagger\beta_k^\dagger - \tilde{\alpha}_k\tilde{\beta}_k - \tilde{\alpha}_k^\dagger\tilde{\beta}_k^\dagger)\sinh 2\theta_k \\
& \quad\times\big\{\sinh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle\,\{\Psi_k(\omega + \epsilon_k^-) - \Psi_k\} \\
& \quad - \cosh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle\,\{\Psi_k(\omega - \epsilon_k^+) - \Psi_k^0\}\big\},
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
|D^{(2)}_{k2}[\omega]\rangle = {} & S\sinh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle\,\{(\Phi_k^+(\omega) - \Phi_k^+(-\,\epsilon_k^-)) + (\Phi_k^-(\omega) - \Phi_k^-(-\,\epsilon_k^-))\} \\
& - S\,(\cosh 2\theta_k - 1)\,(\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle\,\{\Phi_k^+(\omega) - \Phi_k^+(-\,\epsilon_k^-)\} \\
& - S\,(\cosh 2\theta_k + 1)\,(\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle\,\{\Phi_k^-(\omega) - \Phi_k^-(-\,\epsilon_k^-)\} \\
& + \sinh 2\theta_k\cosh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle\,\{\Psi_k(\omega - \epsilon_k^+) - \Psi_k^*\} \\
& + (\cosh^2 2\theta_k + 1)\,(\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle\,\{\Psi_k(\omega + \epsilon_k^-) - \Psi_k^0\} \\
& + (\alpha_k\beta_k + \alpha_k^\dagger\beta_k^\dagger - \tilde{\alpha}_k\tilde{\beta}_k - \tilde{\alpha}_k^\dagger\tilde{\beta}_k^\dagger)\sinh 2\theta_k \\
& \quad\times\big\{\cosh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle\,\{\Psi_k(\omega + \epsilon_k^-) - \Psi_k^0\} \\
& \quad - \sinh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle\,\{\Psi_k(\omega - \epsilon_k^+) - \Psi_k^*\}\big\}.
\end{aligned} \tag{3.11}$$

Here, $\Phi_k^{\pm}(\epsilon)$ are defined by (A.42) and (A.43), and $\Psi_k(\epsilon)$ is defined by

$$\Psi_k(\epsilon) = \phi_k^{zz}(\epsilon) = \int_0^\infty d\tau \sum_\nu g_{2\nu}^2 \langle 1_{\mathtt{R}}|\Delta(R_{k\nu}^\dagger(\tau)R_{k\nu}(\tau))\Delta(R_{k\nu}^\dagger R_{k\nu})|\rho_{\mathtt{R}}\rangle \exp(i\,\epsilon\,\tau), \tag{3.12}$$

with $\Psi_k(\epsilon_k^+ + \epsilon_k^-) = \Psi_k$ and $\Psi_k(0) = \Psi_k^0$, which are defined by (A.44) and (A.54). The lowest-order part $\chi^{(0)}_{S_k^+ S_k^-}(\omega)$ of the transverse magnetic susceptibility $\chi_{S_k^+ S_k^-}(\omega)$ given by (3.6) in the sin-wave approximation, takes the following forms

$$\chi^{(0)}_{S_k^+ S_k^-}(\omega) = \frac{\hbar\gamma^2 S}{2}\int_0^\infty dt\,\langle 1_{\mathtt{S}}|(a_k + b_k^\dagger)\,U(t)\exp(i\,\omega\,t)\big\{i\,\{a_k^\dagger - \tilde{a}_k + b_k - \tilde{b}_k^\dagger\}|\rho_0\rangle$$
$$+ (\cosh\theta_k - \sinh\theta_k)\,\{|D_{k1}^{(2)}[\omega]\rangle/(2\,(\omega-\epsilon_k^+)) + |D_{k2}^{(2)}[\omega]\rangle/(2\,(\omega+\epsilon_k^-))\}\big\}, \tag{3.13a}$$
$$= \frac{\hbar\gamma^2 S}{2}\int_0^\infty dt\,(\cosh\theta_k - \sinh\theta_k)^2\langle 1_{\mathtt{S}}|\{\alpha_k(t) + \beta_k^{\dagger\dagger}(t)\}\exp(i\,\omega\,t)\big\{i\,\{\alpha_k^\dagger - \tilde{\alpha}_k + \beta_k - \tilde{\beta}_k^\dagger\}|\rho_0\rangle$$
$$+ \{|D_{k1}^{(2)}[\omega]\rangle/(2\,(\omega-\epsilon_k^+)) + |D_{k2}^{(2)}[\omega]\rangle/(2\,(\omega+\epsilon_k^-))\}\big\}, \tag{3.13b}$$

where we have used the axioms (A.26), the Heisenberg operators (A.27a), (A.27b) and their tilde conjugates. According to the transformations (A.33a), (A.33b), (A.37a), (A.37b) and their tilde conjugates, the thermal-state conditions (A.36) and their tilde conjugates, the relations (A.34a) and (A.34b), the axioms (A.7) and their tilde conjugates, the forms (A.57a) and (A.57b) of the quasi-particle operators, we have

$$\langle 1_{\mathtt{S}}|\,\alpha_k(t) = Z_k^\alpha(t)^{1/2}\langle 1_{\mathtt{S}}|\,\lambda_k(t) = Z_k^\alpha(0)^{1/2}\exp\{(-\,i\,\epsilon_k^+ - \Gamma_{k+})\,t\,\}\langle 1_{\mathtt{S}}|\lambda_k$$
$$+ Z_k^\beta(0)^{1/2}\Delta_{k-}^*\frac{\exp\{(-\,i\,\epsilon_k^+ - \Gamma_{k+})\,t\} - \exp\{(i\,\epsilon_k^- - \Gamma_{k-}^*)\,t\}}{i\,(\epsilon_k^+ + \epsilon_k^-) + \Gamma_{k+} - \Gamma_{k-}^*}\langle 1_{\mathtt{S}}|\,\tilde{\xi}_k\,,$$
$$= \exp\{(-\,i\,\epsilon_k^+ - \Gamma_{k+})\,t\,\}\langle 1_{\mathtt{S}}|\,\alpha_k$$
$$+ \Delta_{k-}^*\,\frac{\exp\{(-\,i\,\epsilon_k^+ - \Gamma_{k+})\,t\,\} - \exp\{(i\,\epsilon_k^- - \Gamma_{k-}^*)\,t\,\}}{i\,(\epsilon_k^+ + \epsilon_k^-) + \Gamma_{k+} - \Gamma_{k-}^*}\,\langle 1_{\mathtt{S}}|\,\beta_k^\dagger\,, \tag{3.14a}$$
$$\langle 1_{\mathtt{S}}|\,\beta_k^{\dagger\dagger}(t) = Z_k^\beta(t)^{1/2}\langle 1_{\mathtt{S}}|\,\tilde{\xi}_k(t) = Z_k^\beta(0)^{1/2}\exp\{(i\,\epsilon_k^- - \Gamma_{k-}^*)\,t\,\}\langle 1_{\mathtt{S}}|\,\tilde{\xi}_k$$
$$+ Z_k^\alpha(0)^{1/2}\Delta_{k+}\frac{\exp\{(-\,i\,\epsilon_k^+ - \Gamma_{k+})\,t\} - \exp\{(i\,\epsilon_k^- - \Gamma_{k-}^*)\,t\}}{i\,(\epsilon_k^+ + \epsilon_k^-) + \Gamma_{k+} - \Gamma_{k-}^*}\langle 1_{\mathtt{S}}|\lambda_k\,,$$
$$= \exp\{(i\,\epsilon_k^- - \Gamma_{k-}^*)\,t\,\}\langle 1_{\mathtt{S}}|\,\beta_k^\dagger$$
$$+ \Delta_{k+}\,\frac{\exp\{(-\,i\,\epsilon_k^+ - \Gamma_{k+})\,t\,\} - \exp\{(i\,\epsilon_k^- - \Gamma_{k-}^*)\,t\,\}}{i\,(\epsilon_k^+ + \epsilon_k^-) + \Gamma_{k+} - \Gamma_{k-}^*}\,\langle 1_{\mathtt{S}}|\,\alpha_k\,. \tag{3.14b}$$

By virtue of the commutation relations (A.5), the axioms (A.7) and their tilde conjugates, we obtain

$$X_{k1}^\alpha(\omega) = \langle 1_{\mathtt{S}}|\,\alpha_k\,|D_{k1}^{(2)}[\omega]\rangle/(2\,(\omega-\epsilon_k^+)) = X_{k1}^\alpha(\omega)' + i\,X_{k1}^\alpha(\omega)'',$$
$$= \big\{S\,\{(\cosh 2\theta_k + 1)(\Phi_k^+(\omega) - \Phi_k^+(\epsilon_k^+)) + (\cosh 2\theta_k - 1)(\Phi_k^-(\omega) - \Phi_k^-(\epsilon_k^+))\}$$
$$+ (\cosh^2 2\theta_k + 1)(\Psi_k(\omega-\epsilon_k^+) - \Psi_k^0) - \sinh^2 2\theta_k\,(\Psi_k(\omega+\epsilon_k^-) - \Psi_k)\big\}/\{2\,(\omega-\epsilon_k^+)\}, \tag{3.15a}$$
$$X_{k2}^\alpha(\omega) = \langle 1_{\mathtt{S}}|\,\alpha_k\,|D_{k2}^{(2)}[\omega]\rangle/(2\,(\omega+\epsilon_k^-)) = X_{k2}^\alpha(\omega)' + i\,X_{k2}^\alpha(\omega)'',$$
$$= \big\{S\sinh 2\theta_k\,\{(\Phi_k^+(\omega) - \Phi_k^+(-\,\epsilon_k^-)) + (\Phi_k^-(\omega) - \Phi_k^-(-\,\epsilon_k^-))\}$$
$$+ \sinh 2\theta_k\cosh 2\theta_k\,\{(\Psi_k(\omega-\epsilon_k^+) - \Psi_k^*) - (\Psi_k(\omega+\epsilon_k^-) - \Psi_k^0)\}\big\}/\{2\,(\omega+\epsilon_k^-)\}, \tag{3.15b}$$

$$X_{k1}^\beta(\omega) = \langle 1_{\mathtt{S}}|\,\beta_k^\dagger\,|D_{k1}^{(2)}[\omega]\rangle/(2\,(\omega-\epsilon_k^+)) = X_{k1}^\beta(\omega)' + i\,X_{k1}^\beta(\omega)'',$$
$$= \big\{S\sinh 2\theta_k\{(\Phi_k^+(\omega) - \Phi_k^+(\epsilon_k^+)) + (\Phi_k^-(\omega) - \Phi_k^-(\epsilon_k^+))\}$$
$$+ \sinh 2\theta_k\cosh 2\theta_k\,\{(\Psi_k(\omega-\epsilon_k^+) - \Psi_k^0) - (\Psi_k(\omega+\epsilon_k^-) - \Psi_k)\}\big\}/\{2\,(\omega-\epsilon_k^+)\}, \tag{3.16a}$$
$$X_{k2}^\beta(\omega) = \langle 1_{\mathtt{S}}|\,\beta_k^\dagger\,|D_{k2}^{(2)}[\omega]\rangle/(2\,(\omega+\epsilon_k^-)) = X_{k2}^\beta(\omega)' + i\,X_{k2}^\beta(\omega)'',$$
$$= \big\{S\,\{(\cosh 2\theta_k - 1)(\Phi_k^+(\omega) - \Phi_k^+(-\,\epsilon_k^-)) + (\cosh 2\theta_k + 1)(\Phi_k^-(\omega) - \Phi_k^-(-\,\epsilon_k^-))\}$$
$$-\,(\cosh^2 2\theta_k + 1)(\Psi_k(\omega+\epsilon_k^-) - \Psi_k^0) + \sinh^2 2\theta_k\,(\Psi_k(\omega-\epsilon_k^+) - \Psi_k^*)\big\}/\{2\,(\omega+\epsilon_k^-)\}, \tag{3.16b}$$

where we have defined $X_{k1}^\alpha(\omega)$, $X_{k2}^\alpha(\omega)$, $X_{k1}^\beta(\omega)$ and $X_{k2}^\beta(\omega)$, which correspond to the interference terms, are referred as "the corresponding interference terms" and represent the effects of the memory and initial correlation for the spin system and phonon reservoir. Here, $X^{\alpha(\beta)}_{k1(2)}(\omega)'$ and $X^{\alpha(\beta)}_{k1(2)}(\omega)''$ are the real and imaginary parts of $X^{\alpha(\beta)}_{k1(2)}(\omega)$, respectively. By substituting (3.14a) and (3.14b) into (3.13b), and by performing the integration in (3.13b) considering that $\Gamma'_{k\pm}$ are positive for positive $\epsilon^{\pm}_k$ according to (A.60), the transverse susceptibility $\chi^{(0)}_{S^+_k S^-_k}(\omega)$ in the lowest spin-wave approximation can be expressed as

$$\chi^{(0)}_{S^+_k S^-_k}(\omega) = (\hbar\gamma^2 S/2)(\cosh\theta_k - \sinh\theta_k)^2\{\chi^{(0)1}_{k\pm}(\omega) + \chi^{(0)2}_{k\pm}(\omega) + \chi^{(0)3}_{k\pm}(\omega)\}, \tag{3.17}$$

where $\chi^{(0)n}_{k\pm}(\omega)$ $(n=1,2,3)$ are defined by

$$\chi^{(0)1}_{k\pm}(\omega) = \frac{-i - X^{\alpha}_{k1}(\omega)}{i\,(\omega-\epsilon^+_k) - \Gamma_{k+}} + \frac{-\Delta^*_{k-}\,X^{\beta}_{k1}(\omega)}{\{i\,(\omega-\epsilon^+_k) - \Gamma_{k+}\}\{i\,(\omega+\epsilon^-_k) - \Gamma^*_{k-}\}}, \tag{3.18a}$$

$$\chi^{(0)2}_{k\pm}(\omega) = \frac{i - X^{\beta}_{k2}(\omega)}{i\,(\omega+\epsilon^-_k) - \Gamma^*_{k-}} + \frac{-\Delta_{k+}\,X^{\alpha}_{k2}(\omega)}{\{i\,(\omega-\epsilon^+_k) - \Gamma_{k+}\}\{i\,(\omega+\epsilon^-_k) - \Gamma^*_{k-}\}}, \tag{3.18b}$$

$$\chi^{(0)3}_{k\pm}(\omega) = \frac{-X^{\alpha}_{k2}(\omega)}{i\,(\omega-\epsilon^+_k) - \Gamma_{k+}} + \frac{-X^{\beta}_{k1}(\omega)}{i\,(\omega+\epsilon^-_k) - \Gamma^*_{k-}} + \frac{-\Delta_{k+}\{i + X^{\alpha}_{k1}(\omega)\} + \Delta^*_{k-}\{i - X^{\beta}_{k2}(\omega)\}}{\{i\,(\omega-\epsilon^+_k) - \Gamma_{k+}\}\{i\,(\omega+\epsilon^-_k) - \Gamma^*_{k-}\}}, \tag{3.18c}$$

which lead to the real parts $\chi^{(0)n}_{k\pm}(\omega)'$ and the imaginary parts of $\chi^{(0)n}_{k\pm}(\omega)''$ of $\chi^{(0)n}_{k\pm}(\omega)$ $(n=1,2,3)$, as

$$\begin{aligned}\chi^{(0)1}_{k\pm}(\omega)' = {} & \frac{X^{\alpha}_{k1}(\omega)'\,\Gamma'_{k+} - (1 + X^{\alpha}_{k1}(\omega)'')(\omega - \epsilon^+_k - \Gamma''_{k+})}{(\omega - \epsilon^+_k - \Gamma''_{k+})^2 + (\Gamma'_{k+})^2} \\ & + \big\{\{\Delta'_{k-}X^{\beta}_{k1}(\omega)' + \Delta''_{k-}X^{\beta}_{k1}(\omega)''\}\{(\omega - \epsilon^+_k - \Gamma''_{k+})(\omega + \epsilon^-_k + \Gamma''_{k-}) - \Gamma'_{k+}\Gamma'_{k-}\} \\ & + \{\Delta'_{k-}X^{\beta}_{k1}(\omega)'' - \Delta''_{k-}X^{\beta}_{k1}(\omega)'\}\{(\omega - \epsilon^+_k - \Gamma''_{k+})\Gamma'_{k-} + (\omega + \epsilon^-_k + \Gamma''_{k-})\Gamma'_{k+}\}\big\} \\ & /\big\{\{(\omega - \epsilon^+_k - \Gamma''_{k+})^2 + (\Gamma'_{k+})^2\}\{(\omega + \epsilon^-_k + \Gamma''_{k-})^2 + (\Gamma'_{k-})^2\}\big\},\end{aligned} \tag{3.19a}$$

$$\begin{aligned}\chi^{(0)2}_{k\pm}(\omega)' = {} & \frac{X^{\beta}_{k2}(\omega)'\,\Gamma'_{k-} + (1 - X^{\beta}_{k2}(\omega)'')(\omega + \epsilon^-_k + \Gamma''_{k-})}{(\omega + \epsilon^-_k + \Gamma''_{k-})^2 + (\Gamma'_{k-})^2} \\ & + \big\{\{\Delta'_{k+}X^{\alpha}_{k2}(\omega)' - \Delta''_{k+}X^{\alpha}_{k2}(\omega)''\}\{(\omega - \epsilon^+_k - \Gamma''_{k+})(\omega + \epsilon^-_k + \Gamma''_{k-}) - \Gamma'_{k+}\Gamma'_{k-}\} \\ & + \{\Delta'_{k+}X^{\alpha}_{k2}(\omega)'' + \Delta''_{k+}X^{\alpha}_{k2}(\omega)'\}\{(\omega - \epsilon^+_k - \Gamma''_{k+})\Gamma'_{k-} + (\omega + \epsilon^-_k + \Gamma''_{k-})\Gamma'_{k+}\}\big\} \\ & /\big\{\{(\omega - \epsilon^+_k - \Gamma''_{k+})^2 + (\Gamma'_{k+})^2\}\{(\omega + \epsilon^-_k + \Gamma''_{k-})^2 + (\Gamma'_{k-})^2\}\big\},\end{aligned} \tag{3.19b}$$

$$\begin{aligned}\chi^{(0)3}_{k\pm}(\omega)' = {} & \frac{X^{\alpha}_{k2}(\omega)'\,\Gamma'_{k+} - X^{\alpha}_{k2}(\omega)''(\omega - \epsilon^+_k - \Gamma''_{k+})}{(\omega - \epsilon^+_k - \Gamma''_{k+})^2 + (\Gamma'_{k+})^2} + \frac{X^{\beta}_{k1}(\omega)'\,\Gamma'_{k-} - X^{\beta}_{k1}(\omega)''(\omega + \epsilon^-_k + \Gamma''_{k-})}{(\omega + \epsilon^-_k + \Gamma''_{k-})^2 + (\Gamma'_{k-})^2} \\ & + \big\{\{\Delta'_{k+}X^{\alpha}_{k1}(\omega)' - \Delta''_{k+}(1 + X^{\alpha}_{k1}(\omega)'')\}\{(\omega - \epsilon^+_k - \Gamma''_{k+})(\omega + \epsilon^-_k + \Gamma''_{k-}) - \Gamma'_{k+}\Gamma'_{k-}\} \\ & + \{\Delta'_{k+}(1 + X^{\alpha}_{k1}(\omega)'') + \Delta''_{k+}X^{\alpha}_{k1}(\omega)'\}\{(\omega - \epsilon^+_k - \Gamma''_{k+})\Gamma'_{k-} + (\omega + \epsilon^-_k + \Gamma''_{k-})\Gamma'_{k+}\} \\ & + \{\Delta'_{k-}X^{\beta}_{k2}(\omega)' - \Delta''_{k-}(1 - X^{\beta}_{k2}(\omega)'')\}\{(\omega - \epsilon^+_k - \Gamma''_{k+})(\omega + \epsilon^-_k + \Gamma''_{k-}) - \Gamma'_{k+}\Gamma'_{k-}\} \\ & - \{\Delta'_{k-}(1 - X^{\beta}_{k2}(\omega)'') + \Delta''_{k-}X^{\beta}_{k2}(\omega)'\}\{(\omega - \epsilon^+_k - \Gamma''_{k+})\Gamma'_{k-} + (\omega + \epsilon^-_k + \Gamma''_{k-})\Gamma'_{k+}\}\big\} \\ & /\big\{\{(\omega - \epsilon^+_k - \Gamma''_{k+})^2 + (\Gamma'_{k+})^2\}\{(\omega + \epsilon^-_k + \Gamma''_{k-})^2 + (\Gamma'_{k-})^2\}\big\},\end{aligned} \tag{3.19c}$$

$$\begin{aligned}\chi^{(0)1}_{k\pm}(\omega)'' = {} & \frac{X^{\alpha}_{k1}(\omega)'(\omega - \epsilon^+_k - \Gamma''_{k+}) + (1 + X^{\alpha}_{k1}(\omega)'')\Gamma'_{k+}}{(\omega - \epsilon^+_k - \Gamma''_{k+})^2 + (\Gamma'_{k+})^2} \\ & + \big\{\{\Delta'_{k-}X^{\beta}_{k1}(\omega)'' - \Delta''_{k-}X^{\beta}_{k1}(\omega)'\}\{(\omega - \epsilon^+_k - \Gamma''_{k+})(\omega + \epsilon^-_k + \Gamma''_{k-}) - \Gamma'_{k+}\Gamma'_{k-}\} \\ & - \{\Delta'_{k-}X^{\beta}_{k1}(\omega)' + \Delta''_{k-}X^{\beta}_{k1}(\omega)''\}\{(\omega - \epsilon^+_k - \Gamma''_{k+})\Gamma'_{k-} + (\omega + \epsilon^-_k + \Gamma''_{k-})\Gamma'_{k+}\}\big\} \\ & /\big\{\{(\omega - \epsilon^+_k - \Gamma''_{k+})^2 + (\Gamma'_{k+})^2\}\{(\omega + \epsilon^-_k + \Gamma''_{k-})^2 + (\Gamma'_{k-})^2\}\big\},\end{aligned} \tag{3.20a}$$

$$\begin{aligned}\chi^{(0)2}_{k\pm}(\omega)'' = {} & \frac{X^{\beta}_{k2}(\omega)'(\omega + \epsilon^-_k + \Gamma''_{k-}) - (1 - X^{\beta}_{k2}(\omega)'')\Gamma'_{k-}}{(\omega + \epsilon^-_k + \Gamma''_{k-})^2 + (\Gamma'_{k-})^2} \\ & + \big\{\{\Delta'_{k+}X^{\alpha}_{k2}(\omega)'' + \Delta''_{k+}X^{\alpha}_{k2}(\omega)'\}\{(\omega - \epsilon^+_k - \Gamma''_{k+})(\omega + \epsilon^-_k + \Gamma''_{k-}) - \Gamma'_{k+}\Gamma'_{k-}\} \\ & - \{\Delta'_{k+}X^{\alpha}_{k2}(\omega)' - \Delta''_{k+}X^{\alpha}_{k2}(\omega)''\}\{(\omega - \epsilon^+_k - \Gamma''_{k+})\Gamma'_{k-} + (\omega + \epsilon^-_k + \Gamma''_{k-})\Gamma'_{k+}\}\big\} \\ & /\big\{\{(\omega - \epsilon^+_k - \Gamma''_{k+})^2 + (\Gamma'_{k+})^2\}\{(\omega + \epsilon^-_k + \Gamma''_{k-})^2 + (\Gamma'_{k-})^2\}\big\},\end{aligned} \tag{3.20b}$$

$$
\begin{aligned}
\chi_{k\pm}^{(0)3}(\omega)'' = {} & \frac{X_{k2}^{\alpha}(\omega)'(\omega-\epsilon_k^{+}-\Gamma_{k+}'')+X_{k2}^{\alpha}(\omega)''\Gamma_{k+}'}{(\omega-\epsilon_k^{+}-\Gamma_{k+}'')^2+(\Gamma_{k+}')^2}+\frac{X_{k1}^{\beta}(\omega)'(\omega+\epsilon_k^{-}+\Gamma_{k-}'')+X_{k1}^{\beta}(\omega)''\Gamma_{k-}'}{(\omega+\epsilon_k^{-}+\Gamma_{k-}'')^2+(\Gamma_{k-}')^2} \\
& +\{\{\Delta_{k+}'(1+X_{k1}^{\alpha}(\omega)'')+\Delta_{k+}''X_{k1}^{\alpha}(\omega)'\}\{(\omega-\epsilon_k^{+}-\Gamma_{k+}'')(\omega+\epsilon_k^{-}+\Gamma_{k-}'')-\Gamma_{k+}'\Gamma_{k-}'\} \\
& -\{\Delta_{k+}'X_{k1}^{\alpha}(\omega)'-\Delta_{k+}''(1+X_{k1}^{\alpha}(\omega)'')\}\{(\omega-\epsilon_k^{+}-\Gamma_{k+}'')\Gamma_{k-}'+(\omega+\epsilon_k^{-}+\Gamma_{k-}'')\Gamma_{k+}'\} \\
& -\{\Delta_{k-}'(1-X_{k2}^{\beta}(\omega)'')+\Delta_{k-}''X_{k2}^{\beta}(\omega)'\}\{(\omega-\epsilon_k^{+}-\Gamma_{k+}'')(\omega+\epsilon_k^{-}+\Gamma_{k-}'')-\Gamma_{k+}'\Gamma_{k-}'\} \\
& -\{\Delta_{k-}'X_{k2}^{\beta}(\omega)'-\Delta_{k-}''(1-X_{k2}^{\beta}(\omega)'')\}\{(\omega-\epsilon_k^{+}-\Gamma_{k+}'')\Gamma_{k-}'+(\omega+\epsilon_k^{-}+\Gamma_{k-}'')\Gamma_{k+}'\}\} \\
& /\{\{(\omega-\epsilon_k^{+}-\Gamma_{k+}'')^2+(\Gamma_{k+}')^2\}\{(\omega+\epsilon_k^{-}+\Gamma_{k-}'')^2+(\Gamma_{k-}')^2\}\}. \qquad (3.20c)
\end{aligned}
$$

Then, the real part $\chi_{S_k^+S_k^-}^{(0)}(\omega)'$ and imaginary part $\chi_{S_k^+S_k^-}^{(0)}(\omega)''$ of the transverse susceptibility $\chi_{S_k^+S_k^-}^{(0)}(\omega)$ in the lowest spin-wave approximation are given by

$$
\chi_{S_k^+S_k^-}^{(0)}(\omega)' = (\hbar\gamma^2 S/2)(\cosh\theta_k-\sinh\theta_k)^2\{\chi_{k\pm}^{(0)1}(\omega)'+\chi_{k\pm}^{(0)2}(\omega)'+\chi_{k\pm}^{(0)3}(\omega)'\}, \qquad (3.21a)
$$

$$
\chi_{S_k^+S_k^-}^{(0)}(\omega)'' = (\hbar\gamma^2 S/2)(\cosh\theta_k-\sinh\theta_k)^2\{\chi_{k\pm}^{(0)1}(\omega)''+\chi_{k\pm}^{(0)2}(\omega)''+\chi_{k\pm}^{(0)3}(\omega)''\}. \qquad (3.21b)
$$

Since the second terms of $\chi_{k\pm}^{(0)1}(\omega)$ and $\chi_{k\pm}^{(0)2}(\omega)$ given by (3.18a) and (3.18b), and the third term of $\chi_{k\pm}^{(0)3}(\omega)$ given by (3.18c), can be considered to give small contribution in the resonance region, the real part $\chi_{S_k^+S_k^-}^{(0)}(\omega)'$ and imaginary part $\chi_{S_k^+S_k^-}^{(0)}(\omega)''$ of the transverse susceptibility in the lowest spin-wave approximation take approximately the forms

$$
\chi_{S_k^+S_k^-}^{(0)}(\omega)' \cong \frac{\hbar\gamma^2}{2}\left\{\frac{\Xi_k^{\alpha}(\omega)'\,\Gamma_{k+}'-\Xi_k^{\alpha}(\omega)''\,(\omega-\epsilon_k^{+}-\Gamma_{k+}'')}{(\omega-\epsilon_k^{+}-\Gamma_{k+}'')^2+(\Gamma_{k+}')^2}+\frac{\Pi_k^{\beta}(\omega)'\,\Gamma_{k-}'-\Pi_k^{\beta}(\omega)''\,(\omega+\epsilon_k^{-}+\Gamma_{k-}'')}{(\omega+\epsilon_k^{-}+\Gamma_{k-}'')^2+(\Gamma_{k-}')^2}\right\}, \qquad (3.22a)
$$

$$
\chi_{S_k^+S_k^-}^{(0)}(\omega)'' \cong \frac{\hbar\gamma^2}{2}\left\{\frac{\Xi_k^{\alpha}(\omega)''\,\Gamma_{k+}'+\Xi_k^{\alpha}(\omega)'\,(\omega-\epsilon_k^{+}-\Gamma_{k+}'')}{(\omega-\epsilon_k^{+}-\Gamma_{k+}'')^2+(\Gamma_{k+}')^2}+\frac{\Pi_k^{\beta}(\omega)'\,(\omega+\epsilon_k^{-}+\Gamma_{k-}'')+\Pi_k^{\beta}(\omega)''\,\Gamma_{k-}'}{(\omega+\epsilon_k^{-}+\Gamma_{k-}'')^2+(\Gamma_{k-}')^2}\right\}, \qquad (3.22b)
$$

in the resonance region, where we have put as

$$
\Xi_k^{\alpha}(\omega) = \Xi_k^{\alpha}(\omega)'+i\,\Xi_k^{\alpha}(\omega)'' = S\,(\cosh\theta_k-\sinh\theta_k)^2\{i+X_{k1}^{\alpha}(\omega)+X_{k2}^{\alpha}(\omega)\}, \qquad (3.23a)
$$

$$
\Pi_k^{\beta}(\omega) = \Pi_k^{\beta}(\omega)'+i\,\Pi_k^{\beta}(\omega)'' = S\,(\cosh\theta_k-\sinh\theta_k)^2\{-i+X_{k1}^{\beta}(\omega)+X_{k2}^{\beta}(\omega)\}, \qquad (3.23b)
$$

with the real parts $\Xi_k^{\alpha}(\omega)'$, $\Pi_k^{\beta}(\omega)'$ and the imaginary parts $\Xi_k^{\alpha}(\omega)''$, $\Pi_k^{\beta}(\omega)''$ of $\Xi_k^{\alpha}(\omega)$, $\Pi_k^{\beta}(\omega)$.

The power loss of the transversely rotating magnetic-field given by (3.1) is given by $\hbar\gamma|H_k|^2\omega\chi_{S_k^+S_k^-}(\omega)''$ for the anti-ferromagnetic spin system with the wave-number k [24]. When the anti-ferromagnetic system with the wave-number k is in the periodic motion with the frequency ω, the power absorption of the anti-ferromagnetic system is given by $\hbar\gamma|H_k|^2\omega\chi_{S_k^+S_k^-}(\omega)''$. Hereafter, the power absorption of the anti-ferromagnetic system with the wave-number k in the periodic motion with the frequency ω is referred as "$P_k(\omega)$", i.e.,

$$
P_k(\omega) = \hbar\,\gamma\,|H_k|^2\,\omega\,\chi_{S_k^+S_k^-}(\omega)'', \qquad (3.24)
$$

which is expressed in the lowest spin-wave approximation as

$$
P_k^{(0)}(\omega) = \hbar\,\gamma\,|H_k|^2\,\omega\,\chi_{S_k^+S_k^-}^{(0)}(\omega)''. \qquad (3.25)
$$

The line shape of the power absorption $P_k^{(0)}(\omega)$ has two peaks at frequencies $\omega \cong \epsilon_k^{+}+\Gamma_{k+}''$, $-\epsilon_k^{-}-\Gamma_{k-}''$ according to the approximate form (3.22b) for the imaginary part $\chi_{S_k^+S_k^-}^{(0)}(\omega)''$ of the transverse susceptibility in the lowest spin-wave approximation. For positive frequency ω (>0), the resonance frequency $\omega_{\mathtt{R}k}^{\mathtt{P}}$ and the peak-height (height of peak) $H_{\mathtt{R}k}^{\mathtt{P}}$ in the resonance region of the power absorption $P_k^{(0)}(\omega)$ are approximately given by

$$
\omega_{\mathtt{R}k}^{\mathtt{P}} \cong \epsilon_k^{+}+\Gamma_{k+}'', \qquad (3.26)
$$

$$
H_{\mathtt{R}k}^{\mathtt{P}} \cong \hbar^2\,\gamma^3\,|H_k|^2\,\omega_{\mathtt{R}k}^{\mathtt{P}}\,\Xi_k^{\alpha}(\omega_{\mathtt{R}k}^{\mathtt{P}})''/(2\,\Gamma_{k+}'), \qquad (3.27)
$$

with Γ_{k+}' and Γ_{k+}'' given by (A.59a) and (A.59b), according to (3.22b). In order to obtain the approximate formula of the line half-width $\Delta\omega_{\mathtt{R}k}^{\mathtt{P}}$ in the resonance region of the power absorption $P_k^{(0)}(\omega)$, we put as $\Delta\omega_{\mathtt{R}k}^{\mathtt{P}}/2 = x_1\Gamma_{k+}'$ for the first-step approximation of $\Delta\omega_{\mathtt{R}k}^{\mathtt{P}}$, which satisfies

$$
\frac{1}{2}H_{\mathtt{R}k}^{\mathtt{P}} \cong \hbar^2\,\gamma^3\,|H_k|^2\,\frac{\omega_{\mathtt{R}k}^{\mathtt{P}}}{4\,\Gamma_{k+}'}\Xi_k^{\alpha}(\omega_{\mathtt{R}k}^{\mathtt{P}})'' \cong \hbar^2\,\gamma^3\,|H_k|^2\,\frac{\omega_{\mathtt{R}k}^{\mathtt{P}}+x_1\,\Gamma_{k+}'}{2\,(x_1^2+1)\,\Gamma_{k+}'}\{\Xi_k^{\alpha}(\omega_{\mathtt{R}k}^{\mathtt{P}})''+x_1\,\Xi_k^{\alpha}(\omega_{\mathtt{R}k}^{\mathtt{P}})'\}, \qquad (3.28)
$$

where we have approximated $\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})$ with $\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})$ in the right-hand side of the above equation. Equation (3.28) can be rewritten as

$$\{\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})''-2\,\Gamma'_{k+}\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})'\}\,x_1^2-2\,\{\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})'+\Gamma'_{k+}\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})''\}\,x_1-\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})''\cong 0. \tag{3.29}$$

By obtaining the positive solution of the above second-order equation for x_1, the first-step approximation of the line half-width $\Delta\omega^{\mathtt{P}}_{\mathtt{R}k}$ can be derived as

$$\begin{aligned}2\,x_1\Gamma'_{k+}\cong 2\,\Gamma'_{k+}\big\{&\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})'+\Gamma'_{k+}\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})''+\big\{(\omega^{\mathtt{P}}_{\mathtt{R}k})^2\,\{(\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})')^2+(\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})'')^2\}\\&+(\Gamma'_{k+})^2(\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})'')^2\big\}^{1/2}\big\}/\{\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})''-2\,\Gamma'_{k+}\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})'\}.\end{aligned} \tag{3.30}$$

Then, by putting as $\Delta\omega^{\mathtt{P}}_{\mathtt{R}k}/2=x\Gamma'_{k+}$, the approximate formula of the line half-width $\Delta\omega^{\mathtt{P}}_{\mathtt{R}k}$ for the power absorption $P^{(0)}_{k}(\omega)$, can be derived from the equation

$$\hbar^2\,\gamma^3\,|H_k|^2\,\frac{\omega^{\mathtt{P}}_{\mathtt{R}k}}{4\Gamma'_{k+}}\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})''\cong\hbar^2\,\gamma^3\,|H_k|^2\,\frac{\omega^{\mathtt{P}}_{\mathtt{R}k}+x\,\Gamma'_{k+}}{2\,(x^2+1)\,\Gamma'_{k+}}\{\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})''+x\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})'\}, \tag{3.31}$$

which can be rewritten as

$$\begin{aligned}&\big\{\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})''-2\,\Gamma'_{k+}\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})'\big\}\,x^2-2\,\big\{\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})'\\&\quad+\Gamma'_{k+}\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})''\big\}\,x-\omega^{\mathtt{P}}_{\mathtt{R}k}\,\{2\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})''-\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})''\}\cong 0.\end{aligned} \tag{3.32}$$

By obtaining the positive solution of the above second-order equation for x, the approximate formula of the line half-width $\Delta\omega^{\mathtt{P}}_{\mathtt{R}k}$ in the resonance region of the power absorption $P^{(0)}_{k}(\omega)$ can be derived as

$$\begin{aligned}\Delta\omega^{\mathtt{P}}_{\mathtt{R}k}\cong 2\,\Gamma'_{k+}\big\{&\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})'+\Gamma'_{k+}\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})''\\&+\big\{(\omega^{\mathtt{P}}_{\mathtt{R}k})^2\,(\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})')^2+(\Gamma'_{k+})^2\,(\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})'')^2\\&+2\,\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})''\{\Gamma'_{k+}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})'+\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})''\}\\&-2\,\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Gamma'_{k+}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})'\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})''-(\omega^{\mathtt{P}}_{\mathtt{R}k})^2\,(\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})'')^2\big\}^{1/2}\big\}\\&/\{\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})''-2\,\Gamma'_{k+}\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k}+x_1\Gamma'_{k+})'\}.\end{aligned} \tag{3.33}$$

We consider the dynamics of the transverse magnetization with the wave-number k in the stationary state of the anti-ferromagnetic spin system. In the stationay state, $\langle 1_{\mathtt{S}}|\hbar S^{+}_{k}|\rho_1(t)\rangle$ have the form

$$\langle 1_{\mathtt{S}}|\hbar S^{+}_{k}|\rho_1(t)\rangle=(2/\gamma)\,\chi_{S^{+}_{k}S^{-}_{k}}(\omega)\,H_k\exp(-\,i\,\omega\,t),\qquad\qquad (t\to\infty), \tag{3.34}$$

with $|\rho_1(t)\rangle=\langle 1_{\mathtt{R}}|\rho_{\mathtt{T}1}(t)\rangle=|\mathrm{tr}_{\mathtt{R}}\rho_{\mathtt{T}1}(t)\rangle$, where $\rho_{\mathtt{T}1}(t)$ is the first-order part of the density operator $\rho_{\mathtt{T}}(t)$ for the total system in powers of the external driving magnetic-field. The expectation value $M^{x}_{k}(t)$ of the x-component of the magnetization with the wave-number k, can be expressed as

$$M^{x}_{k}(t)=\{\langle 1_{\mathtt{S}}|\hbar S^{+}_{k}|\rho_1(t)\rangle+\langle 1_{\mathtt{S}}|\hbar S^{-}_{k}|\rho_1(t)\rangle\}/2=\mathrm{Re}\,\langle 1_{\mathtt{S}}|\hbar S^{+}_{k}|\rho_1(t)\rangle, \tag{3.35a}$$
$$=(2/\gamma)\big\{(\chi_{S^{+}_{k}S^{-}_{k}}(\omega)H_k)'\cos(\omega\,t)+(\chi_{S^{+}_{k}S^{-}_{k}}(\omega)H_k)''\sin(\omega\,t)\big\}, \tag{3.35b}$$
$$=(2/\gamma)|\chi_{S^{+}_{k}S^{-}_{k}}(\omega)H_k|\sin\{\omega\,t+\delta_k(\omega)\}, \tag{3.35c}$$

where the phase $\delta_k(\omega)$ is defined by

$$\sin\delta_k(\omega)=(\chi_{S^{+}_{k}S^{-}_{k}}(\omega)H_k)'/|\chi_{S^{+}_{k}S^{-}_{k}}(\omega)H_k|,\qquad\qquad \cos\delta_k(\omega)=(\chi_{S^{+}_{k}S^{-}_{k}}(\omega)H_k)''/|\chi_{S^{+}_{k}S^{-}_{k}}(\omega)H_k|. \tag{3.36}$$

The expectation value $M^{y}_{k}(t)$ of the y-component of the magnetization with wave-number k, can be expressed as

$$M^{y}_{k}(t)=\{\langle 1_{\mathtt{S}}|\hbar S^{+}_{k}|\rho_1(t)\rangle-\langle 1_{\mathtt{S}}|\hbar S^{-}_{k}|\rho_1(t)\rangle\}/(2\,i)=\mathrm{Im}\,\langle 1_{\mathtt{S}}|\hbar S^{+}_{k}|\rho_1(t)\rangle, \tag{3.37a}$$
$$=(2/\gamma)\big\{(\chi_{S^{+}_{k}S^{-}_{k}}(\omega)H_k)''\cos(\omega\,t)-(\chi_{S^{+}_{k}S^{-}_{k}}(\omega)H_k)'\sin(\omega\,t)\big\}. \tag{3.37b}$$
$$=(2/\gamma)|\chi_{S^{+}_{k}S^{-}_{k}}(\omega)H_k|\cos\{\omega\,t+\delta_k(\omega)\}. \tag{3.37c}$$

Thus, the expectation values $M^{x}_{k}(t)$ and $M^{y}_{k}(t)$ of the x-component and y-component of the magnetization with the wave-number k oscillate with the frequency ω and the amplitude $A^{\mathtt{M}}_{k}(\omega)$ given by

$$A^{\mathtt{M}}_{k}(\omega)=(2/\gamma)|\chi_{S^{+}_{k}S^{-}_{k}}(\omega)H_k|=(2/\gamma)|H_k|\,|\chi_{S^{+}_{k}S^{-}_{k}}(\omega)|=(2/\gamma)|H_k|\sqrt{(\chi_{S^{+}_{k}S^{-}_{k}}(\omega)')^2+(\chi_{S^{+}_{k}S^{-}_{k}}(\omega)'')^2}, \tag{3.38}$$

which is expressed in the lowest spin-wave approximation as

$$A_k^{\mathtt{M}(0)}(\omega) = (2/\gamma)|H_k| \sqrt{(\chi^{(0)}_{S_k^+S_k^-}(\omega)')^2 + (\chi^{(0)}_{S_k^+S_k^-}(\omega)'')^2}. \tag{3.39}$$

According to the approximate forms (3.22a) and (3.22b) of the real and imaginary parts in the resonance region of the transverse susceptibility $\chi^{(0)}_{S_k^+S_k^-}(\omega)$ in the lowest spin-wave approximation, the amplitude $A_k^{\mathtt{M}(0)}(\omega)$ of the expectation values of the transverse magnetization, which is referred as "the magnetization-amplitude", has two peaks at frequencies $\omega \cong \epsilon_k^+ + \Gamma''_{k+}, -\epsilon_k^- - \Gamma''_{k-}$. Thus, the expectation values $M_k^x(t)$ amd $M_k^y(t)$ of the x-component and y-component of the magnetization with the wave-number k oscillate with the large amplitude $A_k^{\mathtt{M}(0)}(\omega^{\mathtt{M}}_{\mathtt{R}k})$ at the resonance frequency $\omega^{\mathtt{M}}_{\mathtt{R}k}$, which coincides with the resonance frequency $\omega^{\mathtt{P}}_{\mathtt{R}k}$ of the power absorption $P_k^{(0)}(\omega)$ approximately. For positive frequency ω (>0), the resonance frequency $\omega^{\mathtt{M}}_{\mathtt{R}k}$ and the peak-height (height of peak) $H^{\mathtt{M}}_{\mathtt{R}k}$ of the magnetization-amplitude $A_k^{\mathtt{M}(0)}(\omega)$ with the wave-number k are approximately given by

$$\omega^{\mathtt{M}}_{\mathtt{R}k} \cong \epsilon_k^+ + \Gamma''_{k+}, \tag{3.40}$$

$$H^{\mathtt{M}}_{\mathtt{R}k} \cong \hbar\,\gamma\,|H_k|\,\big\{(\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k})')^2 + (\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k})'')^2\big\}^{1/2}/\,\Gamma'_{k+}, \tag{3.41}$$

with Γ'_{k+} and Γ''_{k+} given by (A.59a) and (A.59b). These approximate formulas can be derived by substituting (3.22a) and (3.22b) into (3.39) in the lowest spin-wave approximation. In order to obtain the approximate formula of the line half-width $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}$ in the resonance region of the magnetization-amplitude $A_k^{\mathtt{M}(0)}(\omega)$ with the wave-number k, we put as $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}/2 = y_1\Gamma'_{k+}$ for the first-step approximation of $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}$, which satisfies

$$\begin{aligned}\frac{1}{2}H^{\mathtt{M}}_{\mathtt{R}k} &\cong \hbar\,\gamma\,\frac{|H_k|}{2\,\Gamma'_{k+}}\big\{(\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k})')^2 + (\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k})'')^2\big\}^{1/2},\\ &\cong \hbar\,\gamma\,\frac{|H_k|}{\Gamma'_{k+}}\bigg\{\Big(\frac{\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k})' - y_1\,\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k})''}{y_1^2+1}\Big)^2 + \Big(\frac{\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k})'' + y_1\,\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k})'}{y_1^2+1}\Big)^2\bigg\}^{1/2},\end{aligned} \tag{3.42}$$

where we have approximated $\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k} + y_1\Gamma'_{k+})$ with $\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k})$ in the right-hand side of the above equation. Equation (3.42) gives the positive solution $y_1 \cong \sqrt{3}$. By putting as $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}/2 = y\Gamma'_{k+}$, the approximate formula of the line half-width $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}$ in the resonance region of the magnetization-amplitude, can be derived from the equation

$$\begin{aligned}\hbar\,\gamma\,\frac{|H_k|}{2\,\Gamma'_{k+}}\big\{(\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k})')^2 + (\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k})'')^2\big\}^{1/2} &\cong \hbar\,\gamma\,\frac{|H_k|}{\Gamma'_{k+}}\bigg\{\Big(\frac{\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k} + \sqrt{3}\,\Gamma'_{k+})' - y\,\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k} + \sqrt{3}\,\Gamma'_{k+})''}{y^2+1}\Big)^2\\ &\quad + \Big(\frac{\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k} + \sqrt{3}\,\Gamma'_{k+})'' + y\,\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k} + \sqrt{3}\,\Gamma'_{k+})'}{y^2+1}\Big)^2\bigg\}^{1/2},\end{aligned} \tag{3.43}$$

which can be rewritten as

$$\big\{(\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k})')^2 + (\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k})'')^2\big\}(y^2+1) \cong 4\,\big\{(\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k} + \sqrt{3}\,\Gamma'_{k+})')^2 + (\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k} + \sqrt{3}\,\Gamma'_{k+})'')^2\big\}. \tag{3.44}$$

By obtaining the positive solution of the above equation for y, the approximate formula of the line half-width $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}$ in the resonance region of the magnetization-amplitude, can be derived as

$$\Delta\omega^{\mathtt{M}}_{\mathtt{R}k} \cong 2\,\Gamma'_{k+}\bigg\{4\,\frac{\{\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k} + \sqrt{3}\,\Gamma'_{k+})'\}^2 + \{\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k} + \sqrt{3}\,\Gamma'_{k+})''\}^2}{\{\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k})'\}^2 + \{\Xi_k^\alpha(\omega^{\mathtt{M}}_{\mathtt{R}k})''\}^2} - 1\bigg\}^{1/2}. \tag{3.45}$$

If the relaxation method, in which the Kubo formula [20] is calculated for the physical system interacting with the heat reservoir, is employed [25] in the van Hove limit [39] or in the narrowing limit [40], in which the correlation time τ_c of the heat reservoir is much less than the relaxation time τ_r of the physical system ($\tau_c \ll \tau_r$ or $\tau_c \to 0$), i.e., the Kubo formula [20] is calculated from the second-order TCL equations with no external driving terms in this limit, one obtains the transverse susceptibility [25]

$$\chi^{\mathrm{rv}}_{S_k^+S_k^-}(\omega) = \frac{i}{4}\int_0^\infty dt\,\langle 1_{\mathtt{S}}|\,\gamma\,\hbar\,S_k^+\,U(t)\exp_{\leftarrow}\Big\{-i\int_0^t d\tau\,\hat{\mathcal{H}}_{\mathtt{S}1}(\tau)\Big\}\,\gamma\,(S_k^- - \tilde{S}_k^+)|\rho_0\rangle\exp(i\,\omega\,t), \tag{3.46}$$

which coincides with the ones without the interference thermal state $|D^{(2)}_{S_k^-}[\omega]\rangle$ in the transverse susceptibility $\chi_{S_k^+S_k^-}(\omega)$ given by (3.6) derived employing the TCLE method. That limit neglects the effects of the memory and initial correlation for the spin system and phonon reservoir. Therefore, the above susceptibility $\chi^{\mathrm{rv}}_{S_k^+S_k^-}(\omega)$ is valid for a quickly damped reservoir (the reservoir correlation time $\tau_c \to 0$), but not for a non-quickly damped reservoir, because the influence of motion of the phonon reservoir on the motoin of the spin system is neglected in that limit. The transverse susceptibility $\chi_{S_k^+ S_k^-}(\omega)$ derived employing the TCLE method includes the interference thermal state $|D^{(2)}_{S_k^-}[\omega]\rangle$, which represents the effects of the memory and initial correlation for the spin system and phonon reservoir, i.e., the effects of deviation from the van Hove limit [39] or the narrowing limit [40], and is valid even if the spin system is interacting with a non-quickly damped phonon-reservoir in the region valid for the second-order perturbation approximation. The coincidence of the TCLE method and relaxation method in the second-order approximation for the system-reservoir interaction [25, 34, 35, 36, 37, 38], means that the interference effects, i.e., the effects of the interference terms or the interference thermal state, which are included in the susceptibility derived employing the TCLE method, are the effects of motion of the phonon reservoir which influence the motoin of the spin system. Therefore, the interference effects are considered to increase the power absorption and magnetization-amplitude in the resonance region to excite the phonon reservoir for a non-quickly damped reservoir, because the external driving field excites not only the spin system but also the phonon reservoir for a non-quickly damped reservoir. These are investigated numerically in the following section.

4 Numerical investigation

In the present section, we assume a damped phonon-reservoir model and numerically investigate the power absorption and the magnetization-amplitude (the amplitude of the expectation value of the transverse magnetization) for the anti-ferromagnetic spin system, which is interacting with the phonon reservoir and with the transversely rotating magnetic-field given by (3.1), under an external static magnetic-field in the spin-wave region. We assume that the phonon reservoir consists of a phonon system coupled directly to the spin system and of a reservoir subsystem coupled to the phonon system, where the reservoir subsystem (R-subsystem) is damped quickly, as done in Refs. [21, 22, 41, 42]. Then, the correlation functions of the phonon operators can be derived using the relaxation theory for the phonon system [44, 45, 46], and are assumed to take the forms

$$\sum_{\nu} |\, g_{1\nu}\,|^2 \langle 1_{\mathrm{R}}|\, R^{\dagger}_{k\nu}(t) R_{k\nu}\,|\rho_{\mathrm{R}}\rangle = g_1^2\, \bar{n}(\omega_{\mathrm{R}k}) \exp(i\,\omega_{\mathrm{R}k}\, t - \gamma_{\mathrm{R}k}\, t), \tag{4.1a}$$

$$\sum_{\nu} |\, g_{1\nu}\,|^2 \langle 1_{\mathrm{R}}|\, R_{k\nu}(t) R^{\dagger}_{k\nu}\,|\rho_{\mathrm{R}}\rangle = g_1^2\, \{\bar{n}(\omega_{\mathrm{R}k}) + 1\} \exp(-\,i\,\omega_{\mathrm{R}k}\, t - \gamma_{\mathrm{R}k}\, t), \tag{4.1b}$$

$$\begin{aligned}\sum_{\nu} g_{2\nu}^2 \langle 1_{\mathrm{R}}|\Delta(R^{\dagger}_{k\nu}(t) R_{k\nu}(t))\Delta(R^{\dagger}_{k\nu} R_{k\nu})|\rho_{\mathrm{R}}\rangle &= \sum_{\nu} g_{2\nu}^2 \langle 1_{\mathrm{R}}|\Delta(R^{\dagger}_{k\nu} R_{k\nu})\Delta(R^{\dagger}_{k\nu}(t) R_{k\nu}(t))|\rho_{\mathrm{R}}\rangle,\\ &= g_2^2\, \bar{n}(\omega_{\mathrm{R}k})\{\bar{n}(\omega_{\mathrm{R}k}) + 1\} \exp(-\,2\,\gamma_{\mathrm{R}k}\, t),\end{aligned} \tag{4.1c}$$

with the coupling constants g_1 and g_2 between the spin and phonon, where $\omega_{\mathrm{R}k}$ and $\gamma_{\mathrm{R}k}$ (>0) are, respectively, the characteristic frequency and damping constant of the phonon reservoir. Here, $\bar{n}(\omega_{\mathrm{R}k})$ is given by

$$\bar{n}(\omega_{\mathrm{R}k}) = \{\exp(\beta\hbar\omega_{\mathrm{R}k}) - 1\}^{-1} = \{\exp(\hbar\omega_{\mathrm{R}k}/(k_{\mathrm{B}}T) - 1\}^{-1}. \tag{4.2}$$

The phonon correlation function (4.1c) is real as assumed in Section 2. By using the above correlation functions, $\Phi_k^{\pm}(\epsilon)$ defined by (A.42) and (A.43) can be expressed as [21].

$$\begin{aligned}\Phi_k^{+}(\epsilon) = \Phi_k^{+}(\epsilon)' + i\,\Phi_k^{+}(\epsilon)'' &= \frac{1}{2}\Big\{1 - \exp\Big(\frac{-\,\hbar\,\epsilon}{k_{\mathrm{B}}\,T}\Big)\Big\} \int_0^{\infty} d\tau \sum_{\nu} |g_{1\nu}|^2 \langle 1_{\mathrm{R}}|R_{k\nu}(\tau) R^{\dagger}_{k\nu}|\rho_{\mathrm{R}}\rangle \exp(i\,\epsilon\,\tau),\\ &= \frac{g_1^2}{2}\Big\{1 - \exp\Big(\frac{-\,\hbar\,\epsilon}{k_{\mathrm{B}}\,T}\Big)\Big\} \frac{\bar{n}(\omega_{\mathrm{R}k}) + 1}{(\epsilon - \omega_{\mathrm{R}k})^2 + \gamma_{\mathrm{R}k}^2}\, \{\gamma_{\mathrm{R}k} + i\,(\epsilon - \omega_{\mathrm{R}k})\},\end{aligned} \tag{4.3}$$

$$\begin{aligned}\Phi_k^{-}(\epsilon) = \Phi_k^{-}(\epsilon)' + i\,\Phi_k^{-}(\epsilon)'' &= \frac{1}{2}\Big\{1 - \exp\Big(\frac{-\,\hbar\,\epsilon}{k_{\mathrm{B}}\,T}\Big)\Big\} \int_0^{\infty} d\tau \sum_{\nu} |g_{1\nu}|^2 \langle 1_{\mathrm{R}}|R^{\dagger}_{k\nu}(\tau) R_{k\nu}|\rho_{\mathrm{R}}\rangle \exp(i\,\epsilon\,\tau),\\ &= \frac{g_1^2}{2}\Big\{1 - \exp\Big(\frac{-\,\hbar\,\epsilon}{k_{\mathrm{B}}\,T}\Big)\Big\} \frac{\bar{n}(\omega_{\mathrm{R}k})}{(\epsilon + \omega_{\mathrm{R}k})^2 + \gamma_{\mathrm{R}k}^2}\, \{\gamma_{\mathrm{R}k} + i\,(\epsilon + \omega_{\mathrm{R}k})\},\end{aligned} \tag{4.4}$$

where $\Phi_k^{\pm}(\epsilon)'$ and $\Phi_k^{\pm}(\epsilon)''$ are, respectively, the real part and imaginary part of $\Phi_k^{\pm}(\epsilon)$. We also have for $\Psi_k(\epsilon)$ defined by (3.12), the forms

$$\Psi_k(\epsilon) = \Psi_k(\epsilon)' + i\,\Psi_k(\epsilon)'' = g_2^2\, \frac{\bar{n}(\omega_{\mathrm{R}k})\{\bar{n}(\omega_{\mathrm{R}k}) + 1\}}{-\,i\,\epsilon + 2\gamma_{\mathrm{R}k}} = g_2^2\, \bar{n}(\omega_{\mathrm{R}k})\{\bar{n}(\omega_{\mathrm{R}k}) + 1\} \frac{2\,\gamma_{\mathrm{R}k} + i\,\epsilon}{\epsilon^2 + (2\,\gamma_{\mathrm{R}k})^2}, \tag{4.5}$$

where $\Psi_k(\epsilon)'$ and $\Psi_k(\epsilon)''$ are, respectively, the real part and imaginary part of $\Psi_k(\epsilon)$. For Ψ_k $[= \Psi_k(\epsilon_k^+ + \epsilon_k^-)]$ and Ψ_k^0 $[= \Psi_k(0)]$ defined by (A.44) and (A.54), respectively, we have

$$\Psi_k = \Psi_k(\epsilon_k^+ + \epsilon_k^-) = \Psi_k' + i\,\Psi_k'' = g_2^2\, \bar{n}(\omega_{\mathrm{R}k})\{\bar{n}(\omega_{\mathrm{R}k}) + 1\} \frac{2\,\gamma_{\mathrm{R}k} + i\,(\epsilon_k^+ + \epsilon_k^-)}{(\epsilon_k^+ + \epsilon_k^-)^2 + (2\,\gamma_{\mathrm{R}k})^2}, \tag{4.6}$$

$$\Psi_k^0 = \Psi_k(0) = g_2^2\, \bar{n}(\omega_{\mathrm{R}k})\{\bar{n}(\omega_{\mathrm{R}k}) + 1\}/(2\,\gamma_{\mathrm{R}k}). \tag{4.7}$$

The above expressions given by (4.3) – (4.7) show that $\Phi_k^{\pm}(\epsilon_k^{\pm})'$ is positive for positive $\epsilon_k^{\pm}$ and that $\Psi_k^0 \geq \Psi_k'$. Then, the forms of $\Gamma_{k\pm}'$, $\Gamma_{k\pm}''$, $\Delta_{k\pm}'$ and $\Delta_{k\pm}''$ given by (A.59a) – (A.59d) can be written as

$$\begin{aligned}\Gamma_{k\pm}' &= \frac{g_1^2 S}{4}\Big\{1-\exp\Big(\frac{-\hbar\,\epsilon_k^{\pm}}{k_{\mathrm{B}}T}\Big)\Big\}\Big\{\bar n(\omega_{\mathrm{R}k})+\frac{1}{2}\pm\frac{1}{2}\Big\}\frac{\gamma_{\mathrm{R}k}\cdot(\cosh 2\theta_k\pm 1)}{(\epsilon_k^{\pm}\mp\omega_{\mathrm{R}k})^2+\gamma_{\mathrm{R}k}^2}\\ &+\frac{g_1^2 S}{4}\Big\{1-\exp\Big(\frac{-\hbar\,\epsilon_k^{\pm}}{k_{\mathrm{B}}T}\Big)\Big\}\Big\{\bar n(\omega_{\mathrm{R}k})+\frac{1}{2}\mp\frac{1}{2}\Big\}\frac{\gamma_{\mathrm{R}k}\cdot(\cosh 2\theta_k\mp 1)}{(\epsilon_k^{\pm}\pm\omega_{\mathrm{R}k})^2+\gamma_{\mathrm{R}k}^2}\\ &+\frac{g_2^2\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}}{4\,\gamma_{\mathrm{R}k}}\Big\{1+\frac{(\epsilon_k^{+}+\epsilon_k^{-})^2\cosh^2 2\theta_k+4\gamma_{\mathrm{R}k}^2}{(\epsilon_k^{+}+\epsilon_k^{-})^2+4\gamma_{\mathrm{R}k}^2}\Big\},\end{aligned} \tag{4.8a}$$

$$\begin{aligned}\Gamma_{k\pm}'' &= \frac{g_1^2 S}{4}\Big\{1-\exp\Big(\frac{-\hbar\,\epsilon_k^{\pm}}{k_{\mathrm{B}}\,T}\Big)\Big\}\Big\{\bar n(\omega_{\mathrm{R}k})+\frac{1}{2}\pm\frac{1}{2}\Big\}\frac{(\epsilon_k^{\pm}\mp\omega_{\mathrm{R}k})(\cosh 2\theta_k\pm 1)}{(\epsilon_k^{\pm}\mp\omega_{\mathrm{R}k})^2+\gamma_{\mathrm{R}k}^2}\\ &+\frac{g_1^2 S}{4}\Big\{1-\exp\Big(\frac{-\hbar\,\epsilon_k^{\pm}}{k_{\mathrm{B}}\,T}\Big)\Big\}\Big\{\bar n(\omega_{\mathrm{R}k})+\frac{1}{2}\mp\frac{1}{2}\Big\}\frac{(\epsilon_k^{\pm}\pm\omega_{\mathrm{R}k})(\cosh 2\theta_k\mp 1)}{(\epsilon_k^{\pm}\pm\omega_{\mathrm{R}k})^2+\gamma_{\mathrm{R}k}^2}\\ &-g_2^2\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}\frac{\epsilon_k^{+}+\epsilon_k^{-}}{2\,\{(\epsilon_k^{+}+\epsilon_k^{-})^2+4\gamma_{\mathrm{R}k}^2\}}\sinh^2 2\theta_k\,,\end{aligned} \tag{4.8b}$$

$$\begin{aligned}\Delta_{k\pm}' &= \frac{g_1^2 S}{4}\Big\{1-\exp\Big(\frac{-\hbar\,\epsilon_k^{\pm}}{k_{\mathrm{B}}T}\Big)\Big\}\Big\{\bar n(\omega_{\mathrm{R}k})+\frac{1}{2}\pm\frac{1}{2}\Big\}\frac{\gamma_{\mathrm{R}k}\sinh 2\theta_k}{(\epsilon_k^{\pm}\mp\omega_{\mathrm{R}k})^2+\gamma_{\mathrm{R}k}^2}\\ &+\frac{g_1^2 S}{4}\Big\{1-\exp\Big(\frac{-\hbar\,\epsilon_k^{\pm}}{k_{\mathrm{B}}T}\Big)\Big\}\Big\{\bar n(\omega_{\mathrm{R}k})+\frac{1}{2}\mp\frac{1}{2}\Big\}\frac{\gamma_{\mathrm{R}k}\sinh 2\theta_k}{(\epsilon_k^{\pm}\pm\omega_{\mathrm{R}k})^2+\gamma_{\mathrm{R}k}^2}\\ &+g_2^2\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}\frac{(\epsilon_k^{+}+\epsilon_k^{-})^2\sinh 2\theta_k\cosh 2\theta_k}{4\gamma_{\mathrm{R}k}\{(\epsilon_k^{+}+\epsilon_k^{-})^2+4\gamma_{\mathrm{R}k}^2\}},\end{aligned} \tag{4.8c}$$

$$\begin{aligned}\Delta_{k\pm}'' &= \frac{g_1^2 S}{4}\Big\{1-\exp\Big(\frac{-\hbar\,\epsilon_k^{\pm}}{k_{\mathrm{B}}\,T}\Big)\Big\}\Big\{\bar n(\omega_{\mathrm{R}k})+\frac{1}{2}\pm\frac{1}{2}\Big\}\frac{(\epsilon_k^{\pm}\mp\omega_{\mathrm{R}k})\sinh 2\theta_k}{(\epsilon_k^{\pm}\mp\omega_{\mathrm{R}k})^2+\gamma_{\mathrm{R}k}^2}\\ &+\frac{g_1^2 S}{4}\Big\{1-\exp\Big(\frac{-\hbar\,\epsilon_k^{\pm}}{k_{\mathrm{B}}\,T}\Big)\Big\}\Big\{\bar n(\omega_{\mathrm{R}k})+\frac{1}{2}\mp\frac{1}{2}\Big\}\frac{(\epsilon_k^{\pm}\pm\omega_{\mathrm{R}k})\sinh 2\theta_k}{(\epsilon_k^{\pm}\pm\omega_{\mathrm{R}k})^2+\gamma_{\mathrm{R}k}^2}\\ &-g_2^2\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}\frac{\epsilon_k^{+}+\epsilon_k^{-}}{2\,\{(\epsilon_k^{+}+\epsilon_k^{-})^2+4\gamma_{\mathrm{R}k}^2\}}\sinh 2\theta_k\cosh 2\theta_k\,.\end{aligned} \tag{4.8d}$$

In Appendix C, we give the forms of the corresponding interference terms $X_{k1(2)}^{\alpha(\beta)}(\omega)$ defined by (3.15a) – (3.16b). We consider the case that the phonon reservoir consists of a phonon system of lattice vibration, which has the frequency proportional to the magnitude $|k|$ of the wave number k, and of a reservoir subsystem coupled to the phonon system, where the reservoir subsystem (R-subsystem) is damped quickly. We assume that the characteristic frequency of the phonon reservoir is given by

$$\omega_{\mathrm{R}k} = V\,|k| + \omega_{\mathrm{R}0}, \tag{4.9}$$

where $\omega_{\mathrm{R}0}$ is the characteristic frequency of the phonon reservoir with the wave number $k=0$ and is the frequency shift of the phonon system, which is generated by the motion of the reservoir subsystem coupled to the phonon system. We also assume for consistency with assumptions (4.1a) – (4.1c) that

$$\sum_{\nu} g_{2\nu}\,\langle 1_{\mathrm{R}}|R_{k\nu}^{\dagger}R_{k\nu}|\rho_{\mathrm{R}}\rangle = g_2\,\bar n(\omega_{\mathrm{R}k}). \tag{4.10}$$

Then, the free spin-wave energies $\epsilon_k^{\pm}$ given by (2.20) can be written as

$$\hbar\,\epsilon_k^{\pm} = 2\,z\,\hbar\,J_1\,S\,\Big\{\sqrt{(\zeta+\kappa)^2-\eta_k^2}\pm h_z\Big\}\pm\hbar\,g_2\,\bar n(\omega_{\mathrm{R}k}), \tag{4.11}$$

with η_k, ζ, κ and h_z defined by (2.10). We consider the case that the spin system and phonon reservoir are in the thermal equilibrium state at the initial time $t=0$. The initial values $n_k^{\alpha}(0)$ and $n_k^{\beta}(0)$ are derived in Appendix D and take the following forms

$$\begin{aligned}n_k^{\alpha}(0) &= \bar n(\epsilon_k^{+}) + g_1^2\,S\,(\cosh 2\theta_k+1)\,\{\bar n(\omega_{\mathrm{R}k})-\bar n(\epsilon_k^{+})\}\,\frac{(\epsilon_k^{+}-\omega_{\mathrm{R}k})^2-(\gamma_{\mathrm{R}k})^2}{2\,\{(\epsilon_k^{+}-\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}^2}\\ &\quad+g_1^2\,S\,(\cosh 2\theta_k-1)\,\{\bar n(\epsilon_k^{+})+\bar n(\omega_{\mathrm{R}k})+1\}\,\frac{(\epsilon_k^{+}+\omega_{\mathrm{R}k})^2-(\gamma_{\mathrm{R}k})^2}{2\,\{(\epsilon_k^{+}+\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}^2}\\ &\quad+g_2^2\sinh^2 2\theta_k\,\{\bar n(\epsilon_k^{+})+\bar n(\epsilon_k^{-})+1\}\,\bar n(\omega_{\mathrm{R}k})\,\{\bar n(\omega_{\mathrm{R}k})+1\}\,\frac{(\epsilon_k^{+}+\epsilon_k^{-})^2-4\,(\gamma_{\mathrm{R}k})^2}{\{(\epsilon_k^{+}+\epsilon_k^{-})^2+4\,(\gamma_{\mathrm{R}k})^2\}^2},\end{aligned} \tag{4.12a}$$

$$n^{\beta}_k(0) = \bar{n}(\epsilon^-_k) + g_1^2\, S\,(\cosh 2\theta_k - 1)\big\{\bar{n}(\epsilon^-_k) + \bar{n}(\omega_{\mathtt{R}k}) + 1\big\}\, \frac{(\epsilon^-_k + \omega_{\mathtt{R}k})^2 - (\gamma_{\mathtt{R}k})^2}{2\,\{(\epsilon^-_k + \omega_{\mathtt{R}k})^2 + (\gamma_{\mathtt{R}k})^2\}^2}$$
$$+ g_1^2\, S\,(\cosh 2\theta_k + 1)\big\{\bar{n}(\omega_{\mathtt{R}k}) - \bar{n}(\epsilon^-_k)\big\}\, \frac{(\epsilon^-_k - \omega_{\mathtt{R}k})^2 - (\gamma_{\mathtt{R}k})^2}{2\,\{(\epsilon^-_k - \omega_{\mathtt{R}k})^2 + (\gamma_{\mathtt{R}k})^2\}^2}$$
$$+ g_2^2 \sinh^2 2\theta_k\, \{\bar{n}(\epsilon^+_k) + \bar{n}(\epsilon^-_k) + 1\}\, \bar{n}(\omega_{\mathtt{R}k})\, \{\bar{n}(\omega_{\mathtt{R}k}) + 1\}\, \frac{(\epsilon^+_k + \epsilon^-_k)^2 - 4\,(\gamma_{\mathtt{R}k})^2}{\{(\epsilon^+_k + \epsilon^-_k)^2 + 4\,(\gamma_{\mathtt{R}k})^2\}^2}. \tag{4.12b}$$

We consider an anti-ferromagnetic system of one-dimensional infinite spins interacting with the phonon reservoir. For the case of a regular-interval ranked spin chain, we have

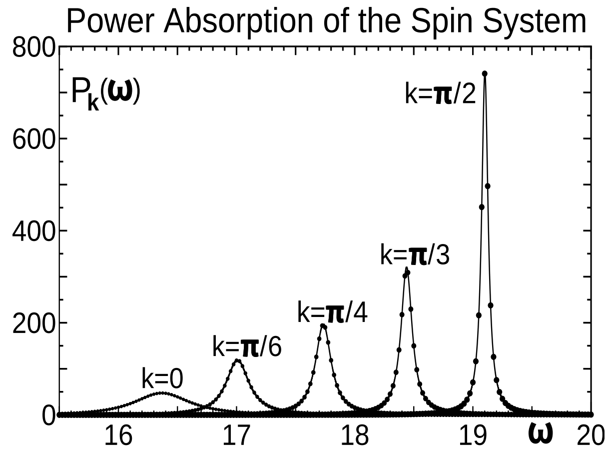
$$z = 2\,, \qquad\qquad \eta_k = \cos k\,, \tag{4.13}$$

where k is the wave number multiplied by the sublattice constant and is referred to as "the wave number" hereafter. We perform the numerical calculations for the case of $g_1/J_1 = 0.25$, $g_2/J_1 = 0.25$, $\omega_{\mathtt{R}0}/J_1 = 0.5$ and $V/J_1 = 0.5$. The damping constant $\gamma_{\mathtt{R}k}$ of the phonon reservoir, which is equal to the inverse of its correlation time τ_c, is assumed to be independent of the wave number k and is taken as $\gamma_{\mathtt{R}k}/J_1 = 0.5$. The wave-number summation is replaced with the integral as

$$\frac{2}{N}\sum_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} dk, \qquad\qquad (N\to\infty), \tag{4.14}$$

for $N\to\infty$, where the wave-number summation goes over $(N/2)$ wave-numbers. The wave-number summation is performed by the numerical integration for $N\to\infty$. In Appendix E, we investigate numerically the region valid for the lowest spin-wave approximation in the anti-ferromagnetic system of one-dimensional infinite spins. In Appendix E, the lowest spin-wave approximation is shown to be valid in the regions of the temperature T and anisotropy energy $\hbar K$ given by $k_{\mathtt{B}}T/(\hbar J_1) \le 1.0$ and $K/J_1 \ge 1.5$, or by $k_{\mathtt{B}}T/(\hbar J_1) \le 1.5$ and $K/J_1 \ge 2.0$, for the spin-magnitude $S \ge 5/2$, $\zeta\ [= J_2/J_1] = 1.0$ and $\omega_z/J_1 = 1.0$, in the meaning that $n^a/(4S)\ [=\langle n_l\rangle/(4S)]$ and $n^b/(4S)\ [=\langle n_m\rangle/(4S)]$, which correspond to the expectation values of the second terms in the expansions given by Eqs. (2.3) and (2.5), respectively, are smaller than about 0.01, where the expectation values $n^a\ [= n^a(\infty)]$ and $n^b\ [= n^b(\infty)]$ are, respectively, the expectation values of the up-spin deviation number and down-spin deviation number in the infinite time limit $(t\to\infty)$.

We next investigate numerically the power absorption and the amplitude of the expectation value of the transverse magnetization, which is referred as "the magnetization-amplitude", for the anti-ferromagnetic spin system in the region valid for the lowest spin-wave approximation, meaning that $n^a/(4S)\ [=\langle n_l\rangle/(4S)]$ and $n^b/(4S)\ [=\langle n_m\rangle/(4S)]$, which correspond to the expectation values of the second terms in the expansions given by Eqs. (2.3) and (2.5), respectively, are smaller than about 0.01. In Appendix F, we give the forms of the collision operator $C^{(2)}$ and the interference thermal state $|D^{(2)}_{S^-_k}[\omega]\rangle$ for the the spin-phonon interaction $\mathcal{H}_{\mathtt{SR}}$ taken in the previous papers [21, 22], which does not reflect the energy transfer between the spin system and phonon reservoir at the sites m of "down" spins. In Fig. 1, the power absorptions $P^{(0)}_k(\omega)$ given by (3.25), scaled by $\hbar^2\gamma^3\,|H_k|^2$, are displayed varying the frequency ω scaled by J_1 from 15.5 to 20.0 for the cases of wave numbers $k = 0, \pi/6, \pi/4, \pi/3, \pi/2$, respectively, and for the spin-magnitudes $S = 5/2$, the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1) = 1.0$ and the anisotropy energy $\hbar K$ given by $K/J_1 = 2.0$, with $\zeta\ [= J_2/J_1] = 1.0$ and $\omega_z/J_1 = 1.0$. In Fig. 2, the magnetization-amplitudes $A^{\mathtt{M}(0)}_k(\omega)$ given by (3.39), scaled by $\hbar\gamma|H_k|/J_1$, are displayed varying the frequency ω scaled by J_1 from 15.5 to 20.0 for the cases of wave numbers $k = 0, \pi/6, \pi/4, \pi/3, \pi/2$, respectively, and for the spin-magnitudes $S = 5/2$, the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1) = 1.0$ and the anisotropy energy $\hbar K$ given by $K/J_1 = 2.0$, with $\zeta\ [= J_2/J_1] = 1.0$ and $\omega_z/J_1 = 1.0$. In Figs. 1 and 2, the results derived according to the spin-phonon interaction given by (2.21), which reflect the energy transfer between the spin system and phonon reservoir, are displayed by the solid lines, and the results derived according to the spin-phonon interaction given by (F.2), which is taken in the previous papers [21, 22] and does not reflect the energy transfer between the spin system and phonon reservoir at the sites m of "down" spins, are displayed by the dots. The latter results coincide well with the former results. Figures 1 and 2 show that the power absorption and magnetization-amplitude have a peak for each wave-number, and that in the resonance regions, as the wave number k becomes large, the resonance frequencies become large, the peak-heights (heights of peak) increase, and the line half-widths decrease. When the external driving magnetic-field is uniform in space, the power absorption and magnetization-amplitude of the spin system in the stationary state are given by $P^{(0)}_k(\omega)$ and $A^{\mathtt{M}(0)}_k(\omega)$ with the wave number $k = 0$ in the lowest spin-wave approximation. In Fig. 3, the power absorptions $P^{(0)}_k(\omega)$ given by (3.25), scaled by $\hbar^2\gamma^3\,|H_k|^2$, are displayed varying the frequency ω scaled by J_1 from 14.0 to 33.0 for the cases of spin-magnitudes $S = 5/2$, 3, 7/2, 4, 9/2, and for the wave-number $k = 0$, the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1) = 1.0$ and the anisotropy energy $\hbar K$ given by $K/J_1 = 2.0$, with $\zeta\ [= J_2/J_1] = 1.0$ and $\omega_z/J_1 = 1.0$. In Fig. 4, the magnetization-amplitudes $A^{\mathtt{M}(0)}_k(\omega)$ given by (3.39), scaled by $\hbar\gamma|H_k|/J_1$, are displayed varying the frequency ω scaled by J_1 from 14.0 to 33.0 for the cases of spin-magnitudes $S = 5/2$, 3, 7/2, 4, 9/2, and for the wave-number $k = 0$, the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1) = 1.0$ and the anisotropy energy $\hbar K$ given by $K/J_1 = 2.0$, with $\zeta\ [= J_2/J_1] = 1.0$ and $\omega_z/J_1 = 1.0$. In Figs. 3 and 4, the results



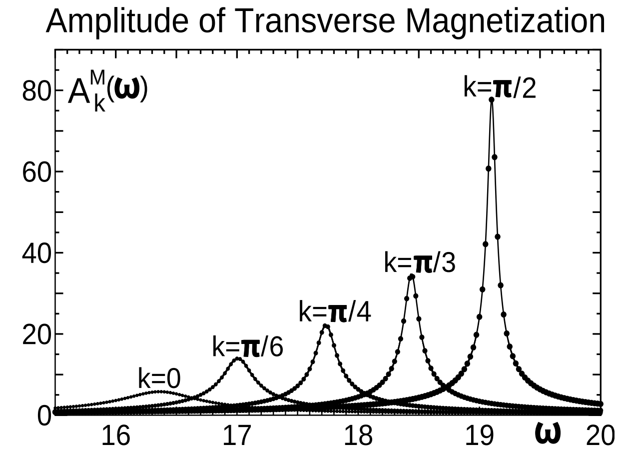
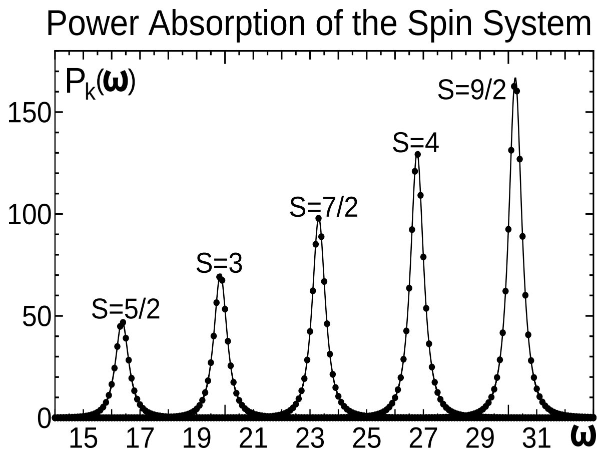


Figure 1: The power absorptions $P_k^{(0)}(\omega)$ given by (3.25), scaled by $\hbar^2\gamma^3\,|H_k|^2$, are displayed varying the frequency ω scaled by J_1 from 15.5 to 20.0 for the cases of wave numbers $k=0,\pi/6,\pi/4,\pi/3,\pi/2$, and for the spin-magnitudes $S=5/2$, the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1)=1.0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with ζ $[=J_2/J_1]=1.0$ and $\omega_z/J_1=1.0$. The results derived according to the spin-phonon interaction given by (2.21) are displayed by the solid lines, and the results derived according to the spin-phonon interaction given by (F.2) are displayed by the dots.

Figure 2: The magnetization-amplitudes $A_k^{\mathtt{M}(0)}(\omega)$ given by (3.39), scaled by $\hbar\gamma|H_k|/J_1$, are displayed varying the frequency ω scaled by J_1 from 15.5 to 20.0 for the cases of wave numbers $k=0,\pi/6,\pi/4,\pi/3,\pi/2$, and for the spin-magnitudes $S=5/2$, the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1)=1.0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with ζ $[=J_2/J_1]=1.0$ and $\omega_z/J_1=1.0$. The results derived according to the spin-phonon interaction given by (2.21) are displayed by the solid lines, and the results derived according to the spin-phonon interaction given by (F.2) are displayed by the dots.

derived according to the spin-phonon interaction given by (2.21), which reflect the energy transfer between the spin system and phonon reservoir, are displayed by the solid lines, and the results derived according to the spin-phonon interaction given by (F.2), which is taken in the previous papers [21, 22] and does not reflect the energy transfer between the spin system and phonon reservoir at the sites m of “down” spins, are displayed by the dots. The latter results coincide well with the former results. Figures 3 and 4 show that in the resonance regions of the power absorption and magnetization-amplitude, as the spin-magnitude S becomes large, the resonance frequencies become large, and the peak-heights increase. As seen in Figs. $1-4$, the facts that the results derived according to the spin-phonon interaction given by (F.2) coincide well with the results derived according to the spin-phonon interaction given by (2.21), show that the energy transfer between the spin system and phonon reservoir at the sites m of “down” spins has few influence on the power absorptions and magnetization-amplitudes. Figs. $1-4$ also show that each peak of the line shapes of magnetization-amplitude $A_k^{\mathtt{M}(0)}(\omega)$ has the hemline longer than that of the power absorption $P_k^{(0)}(\omega)$. Let us see temperature dependence of the line shapes in the resonance regions of the power absorption $P_k^{(0)}(\omega)$ and magnetization-amplitudes $A_k^{\mathtt{M}(0)}(\omega)$. In Fig. 5, we display the resonance frequency $\omega_{\mathtt{R}k}^{\mathtt{P}}$ scaled by J_1 in the resonance region of the power absorption $P_k^{(0)}(\omega)$ varying the temperature T scaled by $\hbar J_1/k_{\mathtt{B}}$ from 0.1 to 1.1 for the cases of spin-magnitudes $S=5/2$, 3, 7/2, 4, 9/2, and for the wave number $k=0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with ζ $[=J_2/J_1]=1.0$ and $\omega_z/J_1=1.0$. The resonance frequency $\omega_{\mathtt{R}k}^{\mathtt{P}}$ investigated calculating numerically the power absorption $P_k^{(0)}(\omega)$ given by (3.25), are displayed by the solid lines, and the approximate formula given by (3.26) for the resonance frequency $\omega_{\mathtt{R}k}^{\mathtt{P}}$ are displayed by the dots. In Fig. 6, we display the resonance frequency $\omega_{\mathtt{R}k}^{\mathtt{M}}$ scaled by J_1 in the resonance region of the magnetization-amplitude $A_k^{\mathtt{M}(0)}(\omega)$ varying the temperature T scaled by $\hbar J_1/k_{\mathtt{B}}$ from 0.1 to 1.1 for the cases of spin-magnitudes $S=5/2$, 3, 7/2, 4, 9/2, and for the wave number $k=0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with ζ $[=J_2/J_1]=1.0$ and $\omega_z/J_1=1.0$. The resonance frequency $\omega_{\mathtt{R}k}^{\mathtt{M}}$ investigated calculating numerically the magnetization-amplitude $A_k^{\mathtt{M}(0)}(\omega)$ given by (3.39), are displayed by the solid lines, and the approximate formula given by (3.40) for the resonance frequency $\omega_{\mathtt{R}k}^{\mathtt{M}}$ are displayed by the dots. Figures 5 and 6 show in the resonance region that as the temperature T becomes high, the resonance frequencies $\omega_{\mathtt{R}k}^{\mathtt{P}}$ and $\omega_{\mathtt{R}k}^{\mathtt{M}}$ become large slightly, that as the spin-magnitude S becomes large, the resonance frequencies $\omega_{\mathtt{R}k}^{\mathtt{P}}$ and $\omega_{\mathtt{R}k}^{\mathtt{M}}$ become large, and that the approximate formulas given by (3.26) and (3.40) for the resonance frequencies $\omega_{\mathtt{R}k}^{\mathtt{P}}$ and $\omega_{\mathtt{R}k}^{\mathtt{M}}$, coincide well with the results investigated calculating numerically the power absorption $P_k^{(0)}(\omega)$ and magnetization-amplitude $A_k^{\mathtt{M}(0)}(\omega)$ given by (3.25) and (3.39) for the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1)\le 1.1$. In Fig. 7, we display the natural logarithm $\log(H_{\mathtt{R}k}^{\mathtt{P}})$ of the peak-height $H_{\mathtt{R}k}^{\mathtt{P}}$ (height of peak) scaled by $\hbar^2\gamma^3\,|H_k|^2$ in the resonance region of the power absorption $P_k^{(0)}(\omega)$ varying the temperature T scaled by $\hbar J_1/k_{\mathtt{B}}$ from 0.1 to 1.1 for the cases of spin-magnitudes



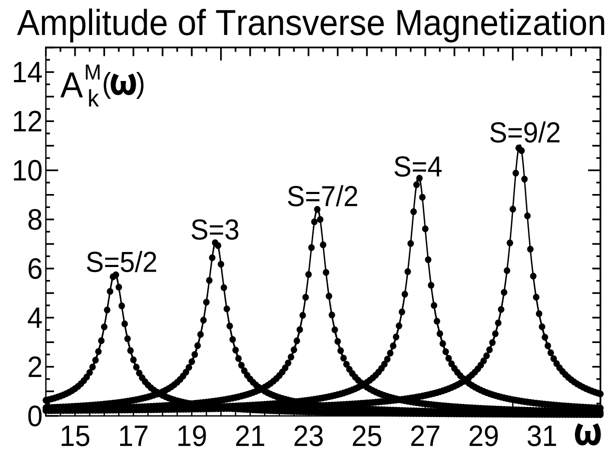
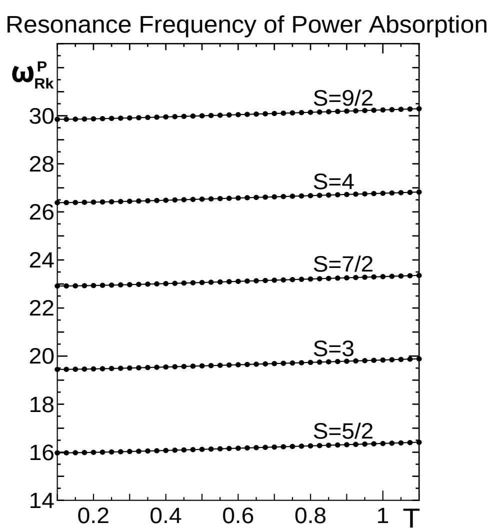


Figure 3: The power absorptions $P_k^{(0)}(\omega)$ given by (3.25), scaled by $\hbar^2\gamma^3\,|H_k|^2$, are displayed varying the frequency ω scaled by J_1 from 14.0 to 33.0 for the cases of spin-magnitudes $S=5/2$, 3, 7/2, 4, 9/2, and for the wave-number $k=0$, the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1)=1.0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$. The results derived according to the spin-phonon interaction given by (2.21) are displayed by the solid lines, and the results derived according to the spin-phonon interaction given by (F.2) are displayed by the dots.

Figure 4: The magnetization-amplitudes $A_k^{\mathtt{M}(0)}(\omega)$ given by (3.39), scaled by $\hbar\gamma|H_k|/J_1$, are displayed varying the frequency ω scaled by J_1 from 14.0 to 33.0 for the cases of spin-magnitudes $S=5/2$, 3, 7/2, 4, 9/2, and for the wave-number $k=0$, the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1)=1.0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$. The results derived according to the spin-phonon interaction given by (2.21) are displayed by the solid lines, and the results derived according to the spin-phonon interaction given by (F.2) are displayed by the dots.



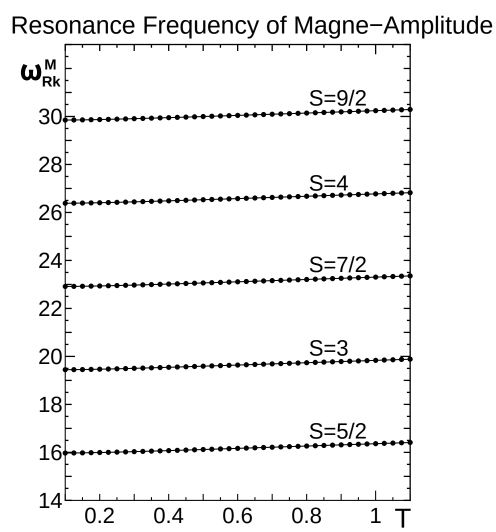
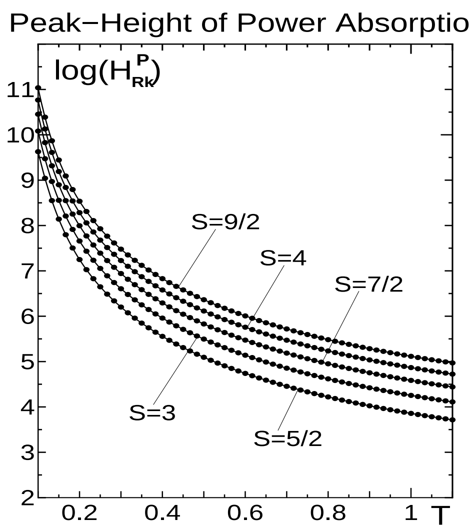


Figure 5: The resonance frequency $\omega_{\mathtt{R}k}^{\mathtt{P}}$ scaled by J_1 for the power absorption $P_k^{(0)}(\omega)$, are displayed by the solid lines varying the temperature T scaled by $\hbar J_1/k_{\mathtt{B}}$ from 0.1 to 1.1 for the cases of spin-magnitudes $S=5/2$, 3, 7/2, 4, 9/2, and for the wave-number $k=0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$. The dots denote the approximate formula given by (3.26) for the resonance frequency $\omega_{\mathtt{R}k}^{\mathtt{P}}$.

Figure 6: The resonance frequency $\omega_{\mathtt{R}k}^{\mathtt{M}}$ scaled by J_1 for the magnetization-amplitudes $A_k^{\mathtt{M}(0)}(\omega)$, are displayed by the solid lines varying the temperature T scaled by $\hbar J_1/k_{\mathtt{B}}$ from 0.1 to 1.1 for the cases of spin-magnitudes $S=5/2$, 3, 7/2, 4, 9/2, and for the wave number $k=0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$. The dots denote the approximate formula given by (3.40) for the resonance frequency $\omega_{\mathtt{R}k}^{\mathtt{M}}$.

$S=5/2$, 3, 7/2, 4, 9/2, and for the wave number $k=0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with ζ $[=J_2/J_1]=1.0$ and $\omega_z/J_1=1.0$. The natural logarithm $\log(H^{\mathtt{P}}_{\mathtt{R}k})$ of the peak-height $H^{\mathtt{P}}_{\mathtt{R}k}$ investigated calculating numerically the power absorption $P^{(0)}_k(\omega)$ given by (3.25), are displayed by the solid lines, and the natural logarithm of the approximate formula given by (3.27) for the peak-height $H^{\mathtt{P}}_{\mathtt{R}k}$ are displayed by the dots. In Fig. 8, we display the natural logarithm $\log(H^{\mathtt{M}}_{\mathtt{R}k})$ of the peak-height $H^{\mathtt{M}}_{\mathtt{R}k}$ scaled by $\hbar\gamma|H_k|/J_1$ in the resonance region of the magnetization-amplitude $A^{\mathtt{M}(0)}_k(\omega)$ varying the temperature T scaled by $\hbar J_1/k_{\mathtt{B}}$ from 0.1 to 1.1 for the cases of spin-magnitudes $S=5/2$, 3, 7/2, 4, 9/2, and for the wave number $k=0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with ζ $[=J_2/J_1]=1.0$ and $\omega_z/J_1=1.0$. The natural logarithm $\log(H^{\mathtt{M}}_{\mathtt{R}k})$ of the peak-height $H^{\mathtt{M}}_{\mathtt{R}k}$ investigated calculating numerically the magnetization-amplitude $A^{\mathtt{M}(0)}_k(\omega)$ given by (3.39), are displayed by the solid lines, and the natural logarithm of the approximate formula given by (3.41) for the peak-height $H^{\mathtt{M}}_{\mathtt{R}k}$ are displayed by the dots. Figures 7 and 8 show in the resonance region that as the temperature T becomes high, the peak-heights $H^{\mathtt{P}}_{\mathtt{R}k}$ and $H^{\mathtt{M}}_{\mathtt{R}k}$ decrease, that as the spin-magnitude S becomes large, the peak-heights $H^{\mathtt{P}}_{\mathtt{R}k}$ and $H^{\mathtt{M}}_{\mathtt{R}k}$ increases, and that the approximate formulas given by (3.27) and (3.41) for the peak-height $H^{\mathtt{P}}_{\mathtt{R}k}$ and $H^{\mathtt{M}}_{\mathtt{R}k}$, coincide well with the results investigated calculating numerically the power absorption $P^{(0)}_k(\omega)$ and magnetization-amplitude $A^{\mathtt{M}(0)}_k(\omega)$ given by (3.25) and (3.39) for the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1)\leq 1.1$. In Fig. 9, we display the line half-width $\Delta\omega^{\mathtt{P}}_{\mathtt{R}k}$ scaled by J_1 in the resonance



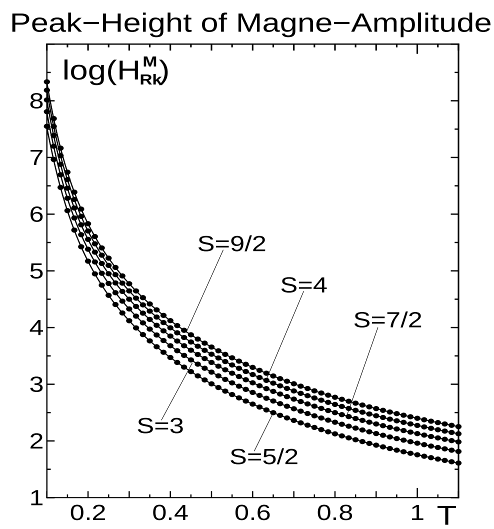
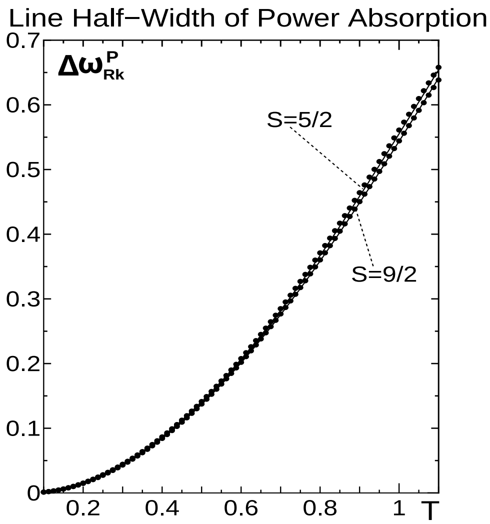


Figure 7: The natural logarithm $\log(H^{\mathtt{P}}_{\mathtt{R}k})$ of the peak-height $H^{\mathtt{P}}_{\mathtt{R}k}$ scaled by $\hbar^2\gamma^3\,|H_k|^2$ for the power absorption $P^{(0)}_k(\omega)$, are displayed by the solid lines varying the temperature T scaled by $\hbar J_1/k_{\mathtt{B}}$ from 0.1 to 1.1 for the cases of spin-magnitudes $S=5/2$, 3, 7/2, 4, 9/2, and for the wave-number $k=0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$. The dots denote the natural logarithm of the approximate formula given by (3.27) for the peak-height $H^{\mathtt{P}}_{\mathtt{R}k}$.

Figure 8: The natural logarithm $\log(H^{\mathtt{M}}_{\mathtt{R}k})$ of the peak-height $H^{\mathtt{M}}_{\mathtt{RF}}$ scaled by $\hbar\gamma|H_k|/J_1$ for the magnetization-amplitudes $A^{\mathtt{M}(0)}_k(\omega)$, are displayed by the solid lines varying the temperature T scaled by $\hbar J_1/k_{\mathtt{B}}$ from 0.1 to 1.1 for the cases of spin-magnitudes $S=5/2$, 3, 7/2, 4, 9/2, and for the wave number $k=0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$. The dots denote the natural logarithm of the approximate formula given by (3.41) for the peak-height $H^{\mathtt{M}}_{\mathtt{R}k}$.

region of the power absorption $P^{(0)}_k(\omega)$ varying the temperature T scaled by $\hbar J_1/k_{\mathtt{B}}$ from 0.1 to 1.1 for the cases of spin-magnitudes $S=5/2$, 9/2, and for the wave number $k=0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with ζ $[=J_2/J_1]=1.0$ and $\omega_z/J_1=1.0$. The line half-width $\Delta\omega^{\mathtt{P}}_{\mathtt{R}k}$ investigated calculating numerically the power absorption $P^{(0)}_k(\omega)$ given by (3.25), are displayed by the solid lines, and the approximate formula given by (3.33) for the line half-width $\Delta\omega^{\mathtt{P}}_{\mathtt{R}k}$ are displayed by the dots. In Fig. 10, we display the line half-width $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}$ scaled by J_1 in the resonance region of the magnetization-amplitude $A^{\mathtt{M}(0)}_k(\omega)$ varying the temperature T scaled by $\hbar J_1/k_{\mathtt{B}}$ from 0.1 to 1.1 for the cases of spin-magnitudes $S=5/2$, 9/2, and for the wave number $k=0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with ζ $[=J_2/J_1]=1.0$ and $\omega_z/J_1=1.0$. The line half-width $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}$ investigated calculating numerically the magnetization-amplitude $A^{\mathtt{M}(0)}_k(\omega)$ given by (3.39), are displayed by the solid lines, and the approximate formula given by (3.45) for the line half-width $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}$ are displayed by the dots. Figures 9 and 10 show in the resonance region that as the temperature T becomes high, the line half-widths $\Delta\omega^{\mathtt{P}}_{\mathtt{R}k}$ and $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}$ increase, that as the spin-magnitude S becomes large, the line half-widths $\Delta\omega^{\mathtt{P}}_{\mathtt{R}k}$ and $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}$ decrease slightly, and that the approximate formulas given by (3.33) and (3.45) for the line half-width $\Delta\omega^{\mathtt{P}}_{\mathtt{R}k}$ and $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}$, coincide well with the results investigated calculating numerically the power absorption $P^{(0)}_k(\omega)$ and magnetization-amplitude $A^{\mathtt{M}(0)}_k(\omega)$ given by (3.25) and (3.39) for the temperature T given by $k_{\mathrm{B}}T/(\hbar J_1) \leq 1.1$. Figures 9 and 10 also show that the line half-widths in the resonance region of the magnetization-amplitude are larger than those of the power absorption.



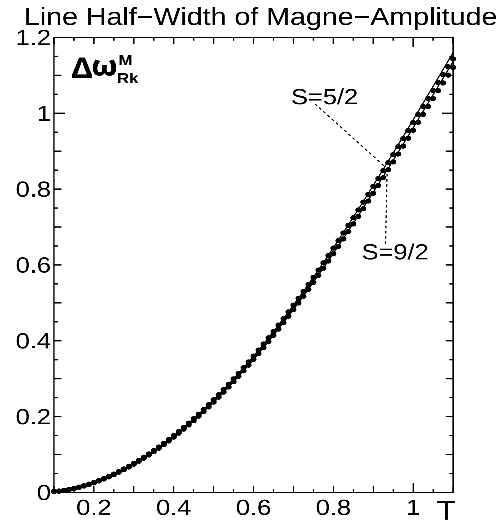


Figure 9: The line half-width $\Delta\omega^{\mathtt{P}}_{\mathtt{R}k}$ scaled by J_1 for the power absorption $P^{(0)}_k(\omega)$, are displayed by the solid lines varying the temperature T scaled by $\hbar J_1/k_{\mathrm{B}}$ from 0.1 to 1.1 for the cases of spin-magnitudes $S=5/2$, $9/2$, and for the wave-number $k=0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$. The dots denote the approximate formula given by (3.33) for the line half-width $\Delta\omega^{\mathtt{P}}_{\mathtt{R}k}$.

Figure 10: The line half-width $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}$ scaled by J_1 for the magnetization-amplitudes $A^{\mathtt{M}(0)}_k(\omega)$, are displayed by the solid lines varying the temperature T scaled by $\hbar J_1/k_{\mathrm{B}}$ from 0.1 to 1.1 for the cases of spin-magnitudes $S=5/2$, $9/2$, and for the wave number $k=0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$. The dots denote the approximate formula given by (3.45) for the line half-width $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}$.

In the last of this section, we investigate the effects of the memory and initial correlation for the anti-ferromagnetic spin system and phonon reservoir numerically. Those effects are represented by the interference terms in the TCLE method and are referred as "the interference effects". In Fig. 11, the power absorption $P^{(0)}_k(\omega)$ scaled by $\hbar^2\gamma^3\,|H_k|^2$ are displayed varying the frequency ω scaled by J_1 from 15.0 to 17.5 in comparison with $P^{\mathrm{rv}(0)}_k(\omega)$ scaled by $\hbar^2\gamma^3\,|H_k|^2$, where $P^{\mathrm{rv}(0)}_k(\omega)$ is the power absorption derived employing the relaxation method [25] in the van Hove limit [39] or in the narrowing limit [40], and is given by

$$P^{\mathrm{rv}(0)}_k(\omega) = \hbar\,\gamma\,|H_k|^2\,\omega\,\chi^{\mathrm{rv}(0)}_{S^+_k S^-_k}(\omega)'', \tag{4.15}$$

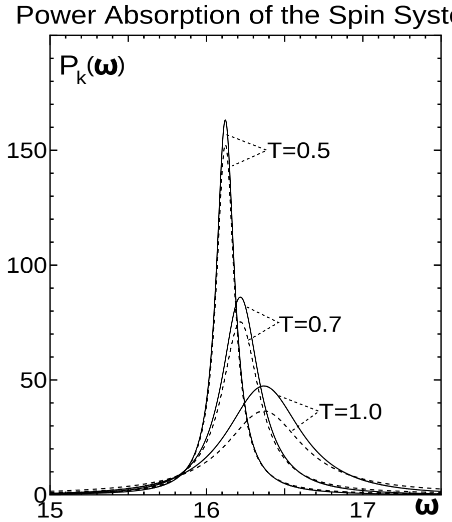
in the lowest spin-wave approximation. Here, $\chi^{\mathrm{rv}(0)}_{S^+_k S^-_k}(\omega)''$ is the imaginary part of the transverse susceptibility $\chi^{\mathrm{rv}(0)}_{S^+_k S^-_k}(\omega)$ derived employing the relaxation method [25] in the van Hove limit [39] or in the narrowing limit [40] in the lowest spin-wave approximation. The transverse susceptibility $\chi^{\mathrm{rv}(0)}_{S^+_k S^-_k}(\omega)$ coincides with the one without the corresponding interference terms $X^{\alpha(\beta)}_{k1(2)}(\omega)$ given by (3.15) and (3.16) in the transverse susceptibility $\chi^{(0)}_{S^+_k S^-_k}(\omega)$ given by (3.13) or (3.17), which has been derived employing the TCLE method in the lowest spin-wave approximation. In Fig. 11, the power absorptions $P^{(0)}_k(\omega)$ and $P^{\mathrm{rv}(0)}_k(\omega)$ are displayed for the cases of temperatures T given by $k_{\mathrm{B}}T/(\hbar J_1)=0.5, 0.7, 1.0$, and for the spin-magnitude $S=5/2$, the wave number $k=0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with $\zeta\ [=J_2/J_1]=1.0$ and $\omega_z/J_1=1.0$. The power absorption $P^{(0)}_k(\omega)$ is displayed by the solid lines and the power absorption $P^{\mathrm{rv}(0)}_k(\omega)$ is displayed by the short dash lines, in Fig. 11. The power absorption $P^{(0)}_k(\omega)$ given by (3.25), which have been derived employing the TCLE method, includes the interference effects which are the effects of the memory and initial correlation for the spin system and phonon reservoir [25], and are neglected in the power absorption $P^{\mathrm{rv}(0)}_k(\omega)$ derived employing the relaxation method [25] in the van Hove limit [39] or in the narrowing limit [40] in the lowest spin-wave approximation. In Fig. 12, the magnetization-amplitude $A^{\mathtt{M}(0)}_k(\omega)$ scaled by $\hbar\gamma|H_k|/J_1$, are displayed varying the frequency ω scaled by J_1 from 15.0 to 17.5 in comparison with $A^{\mathtt{Mrv}(0)}_k(\omega)$ scaled by $\hbar\gamma|H_k|/J_1$, where $A^{\mathtt{Mrv}(0)}_k(\omega)$ is the magnetization-amplitude derived employing the relaxation method [25] in the van Hove limit [39] or in the narrowing limit [40], and is given by

$$A^{\mathtt{Mrv}(0)}_k(\omega) = (2/\gamma)|H_k|\sqrt{(\chi^{\mathrm{rv}(0)}_{S^+_k S^-_k}(\omega)')^2 + (\chi^{\mathrm{rv}(0)}_{S^+_k S^-_k}(\omega)'')^2}, \tag{4.16}$$

in the lowest spin-wave approximation. Here, $\chi^{\rm rv(0)}_{S^+_k S^-_k}(\omega)'$ is the real part of the transverse susceptibility $\chi^{\rm rv(0)}_{S^+_k S^-_k}(\omega)$ derived employing the relaxation method [25] in the van Hove limit [39] or in the narrowing limit [40] in the lowest spin-wave approximation. In Fig. 12, the magnetization-amplitudes $A^{\tt M(0)}_k(\omega)$ and $A^{\tt Mrv(0)}_k(\omega)$ are displayed for the cases of temperatures T given by $k_{\tt B}T/(\hbar J_1) = 0.5, 0.7, 1.0$, and for the spin-magnitude $S = 5/2$, the wave number $k = 0$ and the anisotropy energy $\hbar K$ given by $K/J_1 = 2.0$, with $\zeta\ [= J_2/J_1] = 1.0$ and $\omega_z/J_1 = 1.0$. The magnetization-amplitude $A^{\tt M(0)}_k(\omega)$ is displayed by the solid lines and the magnetization-amplitude $A^{\tt Mrv(0)}_k(\omega)$ are displayed by the short dash lines, in Fig. 12. The magnetization-amplitudes $A^{\tt M(0)}_k(\omega)$ given by (3.39), which have been derived employing the TCLE method, includes the interference effects which are the effects of the memory and initial correlation for the spin system and phonon reservoir [25], and are neglected in the magnetization-amplitude $A^{\tt Mrv(0)}_k(\omega)$ derived employing the relaxation method [25] in the van Hove limit [39] or in the narrowing limit [40] in the lowest spin-wave approximation. Figures 11 and 12 show that the interference effects increase the power absorptions and magnetization-amplitude in the resonance region and produce effects that cannot be disregarded, and that as the temperature T becomes high, those effects become large comparatively. In Fig. 13, the rate $(H^{\tt P}_{{\tt R}k} - H^{\tt Prv}_{{\tt R}k})/H^{\tt P}_{{\tt R}k}$ of the interference effects $(H^{\tt P}_{{\tt R}k} - H^{\tt Prv}_{{\tt R}k})$ for the peak-height $H^{\tt P}_{{\tt R}k}$ of the power absorption $P^{(0)}_k(\omega)$, are displayed varying the temperature T scaled by $\hbar J_1/k_{\tt B}$ from 0.1 to 1.1 for the cases of spin-magnitudes $S = 5/2$, $9/2$, and for the anisotropy energy $\hbar K$ given by $K/J_1 = 2.0$, the wave-number $k = 0$ and the daming constant $\gamma_{{\tt R}k}$ given by $\gamma_{{\tt R}k}/J_1 = 0.5$, with $\zeta\ [= J_2/J_1] = 1.0$ and $\omega_z/J_1 = 1.0$. Here, $H^{\tt P}_{{\tt R}k}$ is the peak-height of the power absorption $P^{(0)}_k(\omega)$, the approximate formula given by (3.27) is used for $H^{\tt P}_{{\tt R}k}$, and $H^{\tt Prv}_{{\tt R}k}$ is the one without the corresponding interference terms $X^{\alpha}_{k1(2)}(\omega)$ in the approximate formula (3.27). In Fig. 14, the rate $(H^{\tt M}_{{\tt R}k} - H^{\tt Mrv}_{{\tt R}k})/H^{\tt M}_{{\tt R}k}$ of the interference effects $(H^{\tt M}_{{\tt R}k} - H^{\tt Mrv}_{{\tt R}k})$ for the peak-height $H^{\tt M}_{{\tt R}k}$ of the magnetization-amplitudes $A^{\tt M(0)}_k(\omega)$, are displayed varying the temperature T scaled by $\hbar J_1/k_{\tt B}$ from 0.1 to 1.1 for the cases of spin-magnitudes $S = 5/2$, $9/2$, and for the anisotropy energy $\hbar K$ given by $K/J_1 = 2.0$, the wave-number $k = 0$ and the daming constant $\gamma_{{\tt R}k}$ given by $\gamma_{{\tt R}k}/J_1 = 0.5$, with $\zeta\ [= J_2/J_1] = 1.0$ and $\omega_z/J_1 = 1.0$. Here, $H^{\tt M}_{{\tt R}k}$ is the peak-height of the magnetization-amplitudes $A^{\tt M(0)}_k(\omega)$, the approximate formula given by (3.41) is used for $H^{\tt M}_{{\tt R}k}$, and $H^{\tt Mrv}_{{\tt R}k}$ is the one without the corresponding interference terms $X^{\alpha}_{k1(2)}(\omega)$ in the approximate formula (3.41). Figures 13 and 14 show in the resonance region that as the temperature T becomes high, the interference effects for the power absorption $P^{(0)}_k(\omega)$ and the magnetization-amplitudes $A^{\tt M(0)}_k(\omega)$, become large. As the spin-magnitude S becomes large, those effects become small slightly. In Fig. 15, the rate $(H^{\tt P}_{{\tt R}k} - H^{\tt Prv}_{{\tt R}k})/H^{\tt P}_{{\tt R}k}$ of the interference effects $(H^{\tt P}_{{\tt R}k} - H^{\tt Prv}_{{\tt R}k})$ for the peak-height $H^{\tt P}_{{\tt R}k}$ of the power absorption $P^{(0)}_k(\omega)$, are displayed varying the daming constant $\gamma_{{\tt R}k}$ scaled by J_1 from 0.5 to 3.5 for the cases of wave numbers $k = 0, \pi/6, \pi/4, \pi/3, \pi/2$, and for the spin-magnitude $S = 5/2$, the temperature T given by $k_{\tt B}T/(\hbar J_1) = 1.0$ and the anisotropy energy $\hbar K$ given by $K/J_1 = 2.0$, with $\zeta\ [= J_2/J_1] = 1.0$ and $\omega_z/J_1 = 1.0$. Here, the peak-height $H^{\tt P}_{{\tt R}k}$ is the peak-height of the power absorption $P^{(0)}_k(\omega)$, the approximate formula given by (3.27) is used for $H^{\tt P}_{{\tt R}k}$, and $H^{\tt Prv}_{{\tt R}k}$ is the one without the corresponding interference terms $X^{\alpha}_{k1(2)}(\omega)$ in the approximate formula (3.27). In Fig. 16, the rate $(H^{\tt M}_{{\tt R}k} - H^{\tt Mrv}_{{\tt R}k})/H^{\tt M}_{{\tt R}k}$ of the interference effects $(H^{\tt M}_{{\tt R}k} - H^{\tt Mrv}_{{\tt R}k})$ for the peak-height $H^{\tt M}_{{\tt R}k}$ of the magnetization-amplitudes $A^{\tt M(0)}_k(\omega)$, are displayed varying the daming constant $\gamma_{{\tt R}k}$ scaled by J_1 from 0.5 to 3.5 for the cases of wave numbers $k = 0, \pi/6, \pi/4, \pi/3, \pi/2$, and for the spin-magnitudes $S = 5/2$, the temperature T given by $k_{\tt B}T/(\hbar J_1) = 1.0$ and the anisotropy energy $\hbar K$ given by $K/J_1 = 2.0$, with $\zeta\ [= J_2/J_1] = 1.0$ and $\omega_z/J_1 = 1.0$. Here, $H^{\tt M}_{{\tt R}k}$ is the peak-height of the magnetization-amplitudes $A^{\tt M(0)}_k(\omega)$, the approximate formula given by (3.41) is used for $H^{\tt M}_{{\tt R}k}$, and $H^{\tt Mrv}_{{\tt R}k}$ is the one without the corresponding interference terms $X^{\alpha}_{k1(2)}(\omega)$ in the approximate formula (3.41). Figures 15 and 16 show in the resonance region that as the damping constant $\gamma_{{\tt R}k}$ of the phonon reservoir becomes small, the interference effects for the power absorption $P^{(0)}_k(\omega)$ and the magnetization-amplitude $A^{\tt M(0)}_k(\omega)$, become large, and also that as the wave number k becomes small, those effects become large. Since the damping constant $\gamma_{{\tt R}k}$ of the phonon reservoir is equal to the inverse of its correlation time τ_c, the interference effects become large as the phonon reservoir is damped slowly. Thus, the interference effects produce effects that cannot be disregarded for the high temperature, for the non-quickly damped reservoir or for the small wave-number.

5 Summary and concluding remarks

We have considered an anti-ferromagnetic spin system with a uniaxial anisotropy energy and an anisotropic exchange interaction under an external static magnetic-field in the spin-wave region, interacting with a phonon reservoir, and have derived a form of the transverse magnetic susceptibility for such a spin system interacting with an external driving magnetic-field, which is a transversely rotating classical field, in the spin-wave approximation by employing the TCLE method of linear response in terms of the non-equilibrium thermo-field dynamics (NETFD), which have been reformulated for the spin-phonon interaction taken to reflect the energy transfer between the spin system and phonon reservoir. We have analytically examined the power absorption and the amplitude of the expectation value of the transverse magnetization, which is referred as "the magnetization-amplitude", for the anti-ferromagnetic spin system, and have derived the approximate formulas of the resonance frequencies, peak-heights (heights of peak) and line half-widths in the resonance region of the power absorption and magnetization-amplitude. We have numerically investigated the power absorption and magnetization-amplitude for an anti-ferromagnetic system of one-dimensional infinite spins



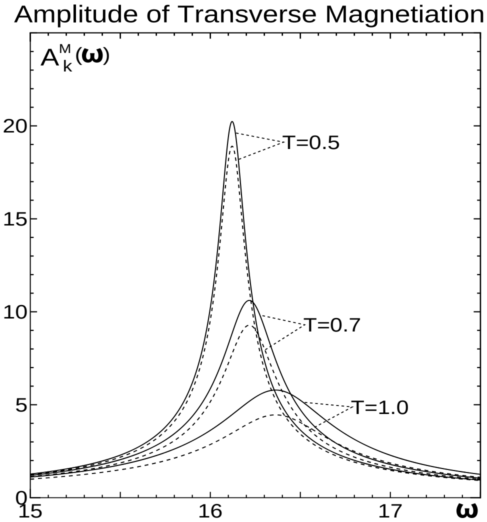
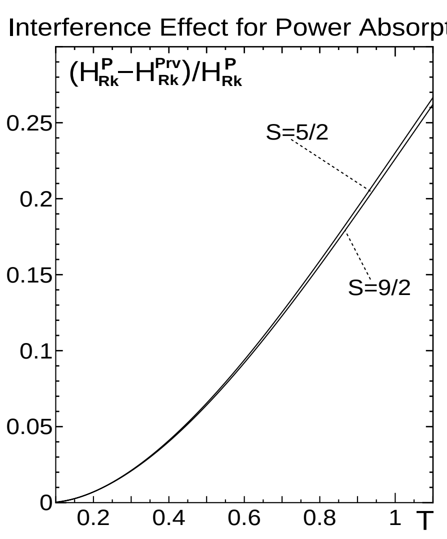


Figure 11: The power absorptions $P_k^{(0)}(\omega)$ and $P_k^{\mathrm{rv}(0)}(\omega)$, scaled by $\hbar^2\gamma^3|H_k|^2$, are displayed varying the frequency ω scaled by J_1 from 15.0 to 17.5, for the cases of temperatures T given by $k_{\mathrm{B}}T/(\hbar J_1)=0.5, 0.7, 1.0$, and for the wave-number $k=0$, the spin-magnitude $S=5/2$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$. The power absorption $P_k^{(0)}(\omega)$ is displayed by the solid lines, and $P_k^{\mathrm{rv}(0)}(\omega)$ is displayed by the short dash lines and coincides with the one without the corresponding interference terms in the power absorption $P_k^{(0)}(\omega)$ derived employing the TCLE method in the lowest spin-wave approximation.

Figure 12: The magnetization-amplitudes $A_k^{\mathtt{M}(0)}(\omega)$ and $A_k^{\mathtt{M}\mathrm{rv}(0)}(\omega)$, scaled by $\hbar\gamma|H_k|/J_1$, are displayed varying the frequency ω scaled by J_1 from 15.0 to 17.5 for the cases of temperatures T given by $k_{\mathrm{B}}T/(\hbar J_1)=0.5, 0.7, 1.0$, and for the wave-number $k=0$, the spin-magnitude $S=5/2$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$. The magnetization-amplitude $A_k^{\mathtt{M}(0)}(\omega)$ is displayed by the solid lines, and $A_k^{\mathtt{M}\mathrm{rv}(0)}(\omega)$ is displayed by the short dash lines and coincides with the one without the corresponding interference terms in the magnetization-amplitude $A_k^{\mathtt{M}(0)}(\omega)$ derived employing the TCLE method in the lowest spin-wave approximation.



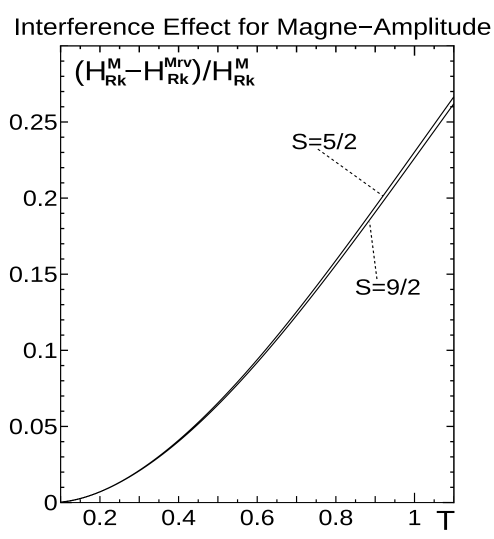
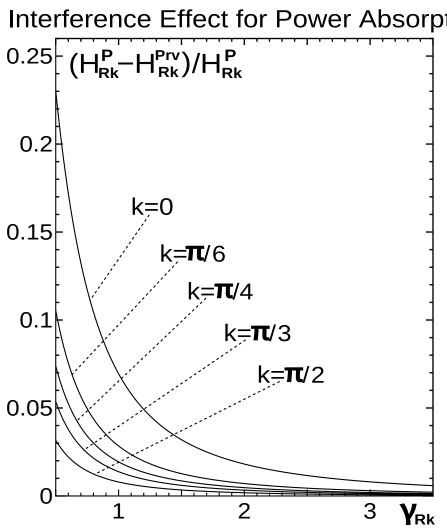


Figure 13: The rate $(H_{\mathtt{R}k}^{\mathtt{P}}-H_{\mathtt{R}k}^{\mathtt{Prv}})/H_{\mathtt{R}k}^{\mathtt{P}}$ of the interference effects $(H_{\mathtt{R}k}^{\mathtt{P}}-H_{\mathtt{R}k}^{\mathtt{Prv}})$ for the peak-height $H_{\mathtt{R}k}^{\mathtt{P}}$ of the power absorption $P_k^{(0)}(\omega)$, are displayed varying the temperature T scaled by $\hbar J_1/k_{\mathtt{B}}$ from 0.1 to 1.1 for the cases of spin-magnitudes $S=5/2$, $9/2$, and for the wave-number $k=0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$. Here, the peak-height $H_{\mathtt{R}k}^{\mathtt{P}}$ is the approximate formula given by (3.27), and $H_{\mathtt{R}k}^{\mathtt{Prv}}$ is the one without the corresponding interference terms $X_{k1(2)}^{\alpha}(\omega)$ in the approximate formula (3.27).

Figure 14: The rate $(H_{\mathtt{R}k}^{\mathtt{M}}-H_{\mathtt{R}k}^{\mathtt{Mrv}})/H_{\mathtt{R}k}^{\mathtt{M}}$ of the interference effects $(H_{\mathtt{R}k}^{\mathtt{M}}-H_{\mathtt{R}k}^{\mathtt{Mrv}})$ for the peak-height $H_{\mathtt{R}k}^{\mathtt{M}}$ of the magnetization-amplitudes $A_k^{\mathtt{M}(0)}(\omega)$, are displayed varying the temperature T scaled by $\hbar J_1/k_{\mathtt{B}}$ from 0.1 to 1.1 for the cases of spin-magnitudes $S=5/2$, $9/2$, and for the wave-number $k=0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$. Here, the peak-height $H_{\mathtt{R}k}^{\mathtt{M}}$ is the approximate formula given by (3.41), and $H_{\mathtt{R}k}^{\mathtt{Mrv}}$ is the one without the corresponding interference terms $X_{k1(2)}^{\alpha}(\omega)$ in the approximate formula (3.41).

by assuming a damped phonon-reservoir model in the region valid for the lowest spin-wave approximation. Here, the valid region means that $n^a/(4S)$ $[=\langle n_l\rangle/(4S)]$ and $n^b/(4S)$ $[=\langle n_m\rangle/(4S)]$, which correspond to the expectation values of the second terms in the expansions given by Eqs. (2.3) and (2.5) respectively, are smaller than about 0.01 in that region, where the expectation values n^a $[=n^a(\infty)]$ and n^b $[=n^b(\infty)]$ are the expectation values of the up-spin deviation number and down-spin deviation number, respectively, in the infinite time limit $(t\rightarrow\infty)$. We have mainly



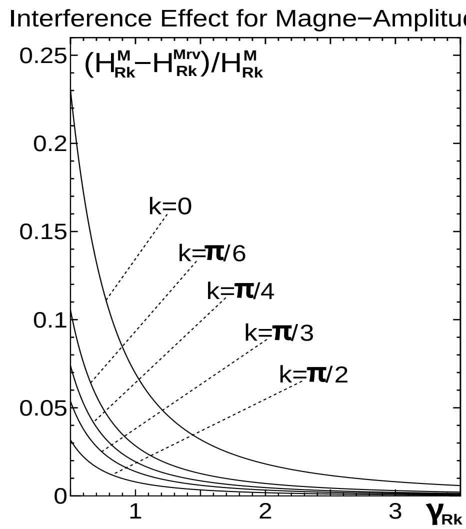


Figure 15: The rate $(H^{\mathtt{P}}_{\mathtt{R}k}-H^{\mathtt{Prv}}_{\mathtt{R}k})/H^{\mathtt{P}}_{\mathtt{R}k}$ of the interference effects $(H^{\mathtt{P}}_{\mathtt{R}k}-H^{\mathtt{Prv}}_{\mathtt{R}k})$ for the peak-height $H^{\mathtt{P}}_{\mathtt{R}k}$ of the power absorption $P^{(0)}_k(\omega)$, are displayed varying the daming constant $\gamma_{\mathtt{R}k}$ of the phonon reservoir, scaled by J_1, from 0.5 to 3.5 for the cases of wave numbers $k=0,\pi/6,\pi/4,\pi/3,\pi/2$, and for the spin-magnitude $S=5/2$, the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1)=1.0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$. Here, the peak-height $H^{\mathtt{P}}_{\mathtt{R}k}$ is the approximate formula given by (3.27), and $H^{\mathtt{Prv}}_{\mathtt{R}k}$ is the one without the corresponding interference terms $X^{\alpha}_{k1(2)}(\omega)$ in the approximate formula (3.27).

Figure 16: The rate $(H^{\mathtt{M}}_{\mathtt{R}k}-H^{\mathtt{Mrv}}_{\mathtt{R}k})/H^{\mathtt{M}}_{\mathtt{R}k}$ of the interference effects $(H^{\mathtt{M}}_{\mathtt{R}k}-H^{\mathtt{Mrv}}_{\mathtt{R}k})$ for the peak-height $H^{\mathtt{M}}_{\mathtt{R}k}$ of the magnetization-amplitudes $A^{\mathtt{M}(0)}_k(\omega)$, are displayed varying the daming constant $\gamma_{\mathtt{R}k}$ of the phonon reservoir, scaled by J_1, for the phonon reservoir from 0.5 to 3.5 for the cases of wave numbers $k=0,\pi/6,\pi/4,\pi/3,\pi/2$, and for the spin-magnitude $S=5/2$, the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1)=1.0$ and the anisotropy energy $\hbar K$ given by $K/J_1=2.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$. Here, the peak-height $H^{\mathtt{M}}_{\mathtt{R}k}$ is the approximate formula given by (3.41), and $H^{\mathtt{Mrv}}_{\mathtt{R}k}$ is the one without the corresponding interference terms $X^{\alpha}_{k1(2)}(\omega)$ in the approximate formula (3.41).

obtained the following results by the numerical investigations for the power absorption and magnetization-amplitude.
1. The power absorption $P^{(0)}_k(\omega)$ and magnetization-amplitude $A^{\mathtt{M}(0)}_k(\omega)$ with the wave number k have a peak for each wave-number. As the wave number k becomes large, the resonance frequencies and peak-heights (heights of peak) increase, and the line half-widths decrease in the resonance region. Thus, as the wave number k becomes large, the line shapes of the power absorption and magnetization-amplitude show "the narrowing" in the resonance region.
2. In the resonance region of the power absorption and magnetization-amplitude, as the spin-magnitude S becomes large, the resonance frequencies become large, the peak-heights increase and the line half-widths decrease slightly.
3. In the resonance region of the power absorption and magnetization-amplitude, as the temperature T becomes high, the resonance frequencies increase slightly, the peak-heights decrease and the line half-widths increase. The approximate formulas of the resonance frequencies, peak-heights and line half-widths, which have been derived in the resonance region of the power absorption and magnetization-amplitude, coincide well with the results investigated calculating numerically the analytic results of the power absorption and magnetization-amplitude.
4. The effects of the memory and initial correlation for the spin system and phonon reservoir, which are represented by the interference terms in the TCLE method and are referred as "the interference effects", increase the power absorption and magnetization-amplitude in the resonance region, and become large as the temperature T becomes high, as the phonon reservoir is damped slowly or as the wave number k becomes small. Thus, the interference effects produce effects that cannot be neglected for the high temperature, for the non-quickly damped reservoir or for the small wave number k. Those effects become small slightly as the spin-magnitude S becomes large.
5. Each peak of the line shapes of magnetization-amplitude has the hemline longer than that of the power absorption. Also, the line half-widths in the resonance region of the magnetization-amplitude are larger than those of the power absorption.
6. The energy transfer between the anti-ferromagnetic spin system and phonon reservoir at the "down"spin sites has few influence on the power absorptions and magnetization-amplitudes of the spin system, i.e., the numerical results derived according to the spin-phonon interaction given by (F.2) coincide almost with those derived according to the

spin-phonon interaction given by (2.21).

We have analytically examined the power absorption and magnetization-amplitude for the anti-ferromagnetic spin system, and have derived the approximate formulas of the resonance frequencies, peak-heights (heights of peak) and line half-widths in the resonance region. The approximate formulas of the resonance frequencies for the power absorption and magnetization-amplitude are given by (3.26) and (3.40), respectively, i.e.

$$\omega^{\mathtt{P}}_{\mathtt{R}k} \cong \epsilon^{+}_{k} + \Gamma''_{k+}, \qquad \omega^{\mathtt{M}}_{\mathtt{R}k} \cong \epsilon^{+}_{k} + \Gamma''_{k+}, \tag{5.1}$$

with Γ''_{k+} given by (A.59b) or (4.8b). As shown in Figs. 5 and 6, the approximate formulas of the resonance frequencies $\omega^{\mathtt{P}}_{\mathtt{R}k}$ and $\omega^{\mathtt{M}}_{\mathtt{R}k}$ coincide well with the results investigated calculating numerically the analytic results of the power absorption $P^{(0)}_{k}(\omega)$ and magnetization-amplitude $A^{\mathtt{M}(0)}_{k}(\omega)$ in the lowest spin-wave approximation, respectively, for the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1) \leq 1.1$. The approximate formulas of the peak-heights for the power absorption and magnetization-amplitude are given by (3.27) and (3.41), respectively, i.e.

$$H^{\mathtt{P}}_{\mathtt{R}k} \cong \hbar^2\,\gamma^3\,|H_k|^2\,\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})''/(2\,\Gamma'_{k+}), \tag{5.2}$$

$$H^{\mathtt{M}}_{\mathtt{R}k} \cong \hbar\gamma\,|H_k|\,\big\{(\Xi^{\alpha}_{k}(\omega^{\mathtt{M}}_{\mathtt{R}k})')^2 + (\Xi^{\alpha}_{k}(\omega^{\mathtt{M}}_{\mathtt{R}k})'')^2\big\}^{1/2}/\,\Gamma'_{k+}, \tag{5.3}$$

with Γ'_{k+} given by (A.59a) or (4.8a), where $\Xi^{\alpha}_{k}(\omega)'$ and $\Xi^{\alpha}_{k}(\omega)''$ are the real and imaginary parts of $\Xi^{\alpha}_{k}(\omega)$ given by (3.23a), respectively, i.e.

$$\Xi^{\alpha}_{k}(\omega)' = S\,(\cosh 2\theta_k - \sinh 2\theta_k)^2\,\{X^{\alpha}_{k1}(\omega)' + X^{\alpha}_{k2}(\omega)'\}, \tag{5.4a}$$

$$\Xi^{\alpha}_{k}(\omega)'' = S\,(\cosh 2\theta_k - \sinh 2\theta_k)^2\,\{1 + X^{\alpha}_{k1}(\omega)'' + X^{\alpha}_{k2}(\omega)''\}. \tag{5.4b}$$

The approximate formulas of the peak-heights $H^{\mathtt{P}}_{\mathtt{R}k}$ and $H^{\mathtt{M}}_{\mathtt{R}k}$ include the real and imaginary parts of the corresponding interference terms $X^{\alpha}_{k1}(\omega)$ and $X^{\alpha}_{k2}(\omega)$ given by (C.3b) and (C.4b) at the resonance frequencies. The interference terms produce the effects that increase the peak-heights of the power absorption and magnetization-amplitude in the resonance region, as seen in Figs. 11 and 12. As shown in Figs. 7 and 8, the approximate formulas of the peak-heights $H^{\mathtt{P}}_{\mathtt{R}k}$ and $H^{\mathtt{M}}_{\mathtt{R}k}$ coincide well with the results investigated calculating numerically the analytic results of the power absorption $P^{(0)}_{k}(\omega)$ and magnetization-amplitude $A^{\mathtt{M}(0)}_{k}(\omega)$ in the lowest spin-wave approximation, respectively, for the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1) \leq 1.1$. The approximate formulas derived for the line half-widths in the resonance region of the power absorption and magnetization-amplitude are given by (3.33) and (3.45), respectively, i.e.,

$$\begin{aligned}\Delta\omega^{\mathtt{P}}_{\mathtt{R}k} \cong 2\,\Gamma'_{k+}\big\{&\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k} + x_1\Gamma'_{k+})' + \Gamma'_{k+}\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k} + x_1\Gamma'_{k+})'' \\ &+ \big\{(\omega^{\mathtt{P}}_{\mathtt{R}k})^2\,(\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k} + x_1\Gamma'_{k+})')^2 + (\Gamma'_{k+})^2\,(\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k} + x_1\Gamma'_{k+})'')^2 \\ &\;+ 2\,\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})''\{\Gamma'_{k+}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k} + x_1\Gamma'_{k+})' + \omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k} + x_1\Gamma'_{k+})''\} \\ &\;- 2\,\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Gamma'_{k+}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k} + x_1\Gamma'_{k+})'\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k} + x_1\Gamma'_{k+})'' - (\omega^{\mathtt{P}}_{\mathtt{R}k})^2\,(\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})'')^2\big\}^{1/2}\big\} \\ &\;/\{\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})'' - 2\,\Gamma'_{k+}\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k} + x_1\Gamma'_{k+})'\},\end{aligned} \tag{5.5}$$

$$\Delta\omega^{\mathtt{M}}_{\mathtt{R}k} \cong 2\,\Gamma'_{k+}\Big\{4\,\frac{\{\Xi^{\alpha}_{k}(\omega^{\mathtt{M}}_{\mathtt{R}k} + \sqrt{3}\Gamma'_{k+})'\}^2 + \{\Xi^{\alpha}_{k}(\omega^{\mathtt{M}}_{\mathtt{R}k} + \sqrt{3}\Gamma'_{k+})''\}^2}{\{\Xi^{\alpha}_{k}(\omega^{\mathtt{M}}_{\mathtt{R}k})'\}^2 + \{\Xi^{\alpha}_{k}(\omega^{\mathtt{M}}_{\mathtt{R}k})''\}^2} - 1\Big\}^{1/2}, \tag{5.6}$$

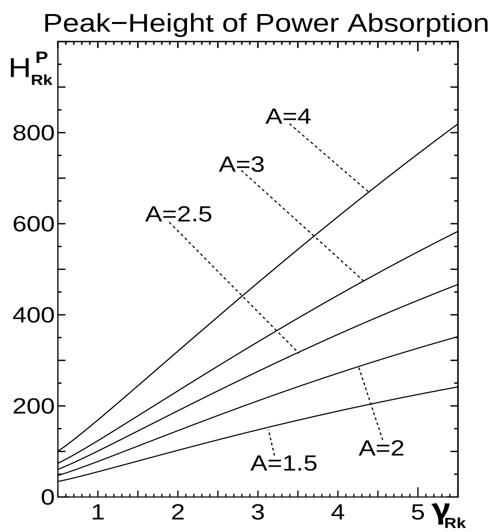
where x_1 is given by

$$\begin{aligned}x_1 \cong \big\{&\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})' + \Gamma'_{k+}\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})'' + \big\{(\omega^{\mathtt{P}}_{\mathtt{R}k})^2\,\{(\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})')^2 + (\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})'')^2\} \\ &+ (\Gamma'_{k+})^2\,(\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})'')^2\big\}^{1/2}\big\}/\{\omega^{\mathtt{P}}_{\mathtt{R}k}\,\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})'' - 2\,\Gamma'_{k+}\Xi^{\alpha}_{k}(\omega^{\mathtt{P}}_{\mathtt{R}k})'\}.\end{aligned} \tag{5.7}$$

As shown in Figs. 9 and 10, the approximate formulas of the line half-widths $\Delta\omega^{\mathtt{P}}_{\mathtt{R}k}$ and $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}$ coincide well with the results investigated calculating numerically the analytic results of the power absorption $P^{(0)}_{k}(\omega)$ and magnetization-amplitude $A^{\mathtt{M}(0)}_{k}(\omega)$ in the lowest spin-wave approximation, respectively, for the temperature given by $k_{\mathtt{B}}T/(\hbar J_1) \leq 1.1$.

The above approximate formulas derived for the resonance frequencies, peak-heights and line half-widths in the resonance region of the power absorption and magnetization-amplitude, are useful for investigating dependence of the line shapes on variation of various physical quantities. As examples, we investigate dependence of the peak-heights and line half-widths in the resonance region of the power absorption and magnetization-amplitude on the anisotropy energy and the damping constant of the phonon reservoir. In Fig. 17, the approximate formula (3.27) or (5.2) of the peak-height $H^{\mathtt{P}}_{\mathtt{R}k}$ in the resonance region of the power absorption $P^{(0)}_{k}(\omega)$, scaled by $\hbar^2\gamma^3\,|H_k|^2$, is displayed varying the daming constant $\gamma_{\mathtt{R}k}$ of the phonon reservoir, scaled by J_1, from 0.5 to 5.5 for the cases of anisotropy energies $\hbar K$ given by $A = K/J_1 = 1.5, 2.0, 2.5, 3.0, 4.0$, and for the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1) = 1.0$ and the spin-magnitude $S = 5/2$, with $\zeta\ [= J_2/J_1] = 1.0$ and $\omega_z/J_1 = 1.0$. In Fig. 18, the approximate formula (3.41) or (5.3) of peak-height $H^{\mathtt{M}}_{\mathtt{R}k}$ in the resonance region of the magnetization-amplitudes $A^{\mathtt{M}(0)}_{k}(\omega)$, scaled by $\hbar\gamma\,|H_k|/J_1$, is displayed varying the daming constant $\gamma_{\mathtt{R}k}$ of the phonon reservoir, scaled by J_1, from 0.5 to 5.5 for the cases of anisotropy energies $\hbar K$

given by $A = K/J_1 = 1.5, 2.0, 2.5, 3.0, 4.0$, and for the temperature T given by $k_{\mathrm{B}}T/(\hbar J_1) = 1.0$ and the spin-magnitude $S = 5/2$, with ζ $[= J_2/J_1] = 1.0$ and $\omega_z/J_1 = 1.0$. The anisotropy energy is denoted as "A" $[= K/J_1]$ in Figs. 17 and 18. Figures 17 and 18 show in the resonance region of the power absorption and magnetization-amplitude that as the damping constant $\gamma_{\mathrm{R}k}$ of the phonon reservoir becomes large, the peak-heights $H^{\mathrm{P}}_{\mathrm{R}k}$ and $H^{\mathrm{M}}_{\mathrm{R}k}$ increase, and also that as the anisotropy energy $\hbar K$ increases, the peak-heights $H^{\mathrm{P}}_{\mathrm{R}k}$ and $H^{\mathrm{M}}_{\mathrm{R}k}$ increase. In Fig. 19, the approximate formula (3.33) or (5.5) of the line half-width $\Delta\omega^{\mathrm{P}}_{\mathrm{R}k}$ in the resonance region of the power absorption $P^{(0)}_k(\omega)$, scaled by J_1, is displayed varying the daming constant $\gamma_{\mathrm{R}k}$ of the phonon reservoir, scaled by J_1, from 0.5 to 5.5 for the cases of anisotropy energies $\hbar K$ given by $A = K/J_1 = 1.5, 2.0, 3.0, 5.0$, and for the temperature T given by $k_{\mathrm{B}}T/(\hbar J_1) = 1.0$ and the spin-magnitude $S = 5/2$, with ζ $[= J_2/J_1] = 1.0$ and $\omega_z/J_1 = 1.0$. In Fig. 20, the approximate formula (3.45) or (5.6) of the line half-width $\Delta\omega^{\mathrm{M}}_{\mathrm{R}k}$ in the resonance region of the magnetization-amplitudes $A^{\mathrm{M}(0)}_k(\omega)$, scaled by J_1, is displayed varying the daming constant $\gamma_{\mathrm{R}k}$ of the phonon reservoir, scaled by J_1, from 0.5 to 5.5 for the cases of anisotropy energies $\hbar K$ given by $A = K/J_1 = 1.5, 2.0, 3.0, 5.0$, and for the temperature T given by $k_{\mathrm{B}}T/(\hbar J_1) = 1.0$ and the spin-magnitude $S = 5/2$, with ζ $[= J_2/J_1] = 1.0$ and $\omega_z/J_1 = 1.0$. The anisotropy energy is denoted as "A" $[= K/J_1]$ in Figs. 19 and 20. Figures 19 and 20 show in the resonance region of the power absorption and magnetization-amplitude that as the damping constant $\gamma_{\mathrm{R}k}$ of the phonon reservoir becomes large, the line half-widths $\Delta\omega^{\mathrm{P}}_{\mathrm{R}k}$ and $\Delta\omega^{\mathrm{M}}_{\mathrm{R}k}$ decrease, and that as the anisotropy energy $\hbar K$ increases, the line half-widths $\Delta\omega^{\mathrm{P}}_{\mathrm{R}k}$ and $\Delta\omega^{\mathrm{M}}_{\mathrm{R}k}$ decrease slightly. Figures 17 − 20 show in the resonance region of the power absorption and magnetization-amplitude that as the damping constant $\gamma_{\mathrm{R}k}$ of the phonon reservoir becomes large, the peak-heights $H^{\mathrm{P}}_{\mathrm{R}k}$ and $H^{\mathrm{M}}_{\mathrm{R}k}$ increase and the line half-widths $\Delta\omega^{\mathrm{P}}_{\mathrm{R}k}$ and $\Delta\omega^{\mathrm{M}}_{\mathrm{R}k}$ decrease. Since the damping constant $\gamma_{\mathrm{R}k}$ of the phonon reservoir is equal to the inverse of its correlation time τ_c, the phonon reservoir is damped quickly as the damping constant become large. Thus, as the phonon reservoir is damped quickly, the line shapes of the power absorption and magnetization-amplitude show "the narrowing", in the resonance region.



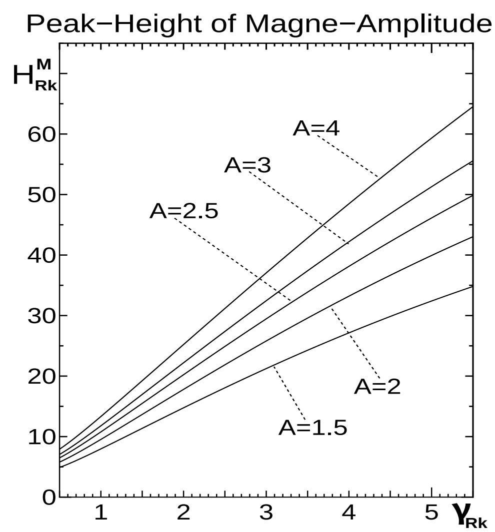
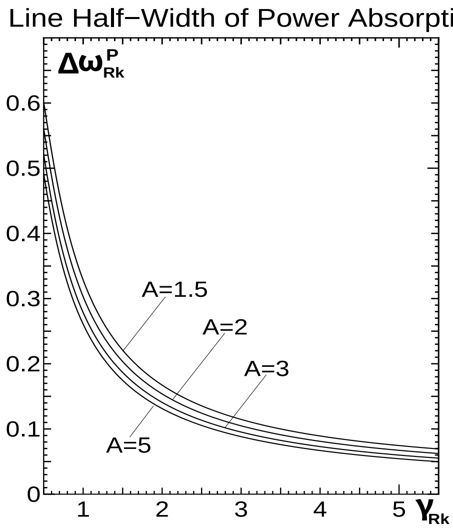


Figure 17: The approximate formula (3.27) for the peak-height $H^{\mathrm{P}}_{\mathrm{R}k}$ in the resonance region of the power absorption $P^{(0)}_k(\omega)$, scaled by $\hbar^2\gamma^3\,|H_k|^2$, is displayed varying the daming constant $\gamma_{\mathrm{R}k}$ of the phonon reservoir, scaled by J_1, from 0.5 to 5.5 for the cases of anisotropy energies $\hbar K$ given by $A = K/J_1 = 1.5, 2.0, 2.5, 3.0, 4.0$, and for the temperature T given by $k_{\mathrm{B}}T/(\hbar J_1) = 1.0$ and the spin-magnitude $S = 5/2$, with $J_2/J_1 = 1.0$ and $\omega_z/J_1 = 1.0$.

Figure 18: The approximate formula (3.41) for the peak-height $H^{\mathrm{M}}_{\mathrm{R}k}$ in the resonance region of the magnetization-amplitudes $A^{\mathrm{M}(0)}_k(\omega)$, scaled by $\hbar\gamma\,|H_k|/J_1$, is displayed varying the daming constant $\gamma_{\mathrm{R}k}$ of the phonon reservoir, scaled by J_1, from 0.5 to 5.5 for the cases of anisotropy energies $\hbar K$ given by $A = K/J_1 = 1.5, 2.0, 2.5, 3.0, 4.0$, and for the temperature T given by $k_{\mathrm{B}}T/(\hbar J_1) = 1.0$ and the spin-magnitude $S = 5/2$, with $J_2/J_1 = 1.0$ and $\omega_z/J_1 = 1.0$.

We have discussed the linear response of an anti-ferromagnetic spin system interacting with a phonon reservoir to an external driving magnetic-field, which is a transversely rotating classical field, by employing the TCLE method in the second-order approximation for the system-reservoir interaction, including the effects of the memory and initial correlation for the spin system and phonon reservoir, i.e., the interference effects (the effects of interference between the external driving field and the phonon reservoir), which are represented by the interference terms or the interference thermal state in the TCLE method, give the effects of the deviation from the van Hove limit [39] or the narrowing limit [40]. The interference effects are the effects of collision of the spin system excited by the external driving field with the phonon reservoir, and influence the motoin of the spin system according to the motion of the phonon reservoir, and therefore those effects increases the power absorption and magnetization-amplitude in the resonance region for a non-quickly damped phonon-reservoir as seen in Figs. 11 and 12, because the external driving field excites not only the spin system but also the phonon reservoir in that region. The interference effects become large as the temperature



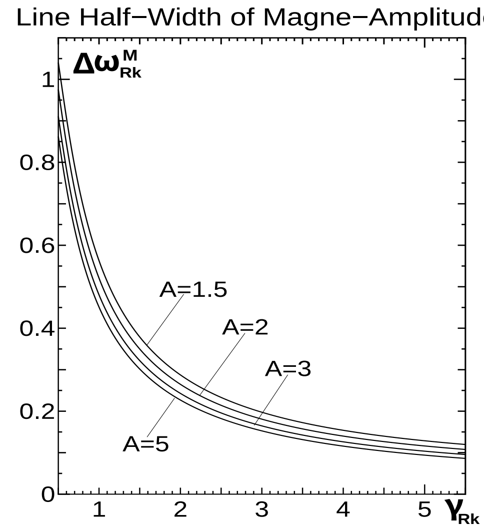


Figure 19: The approximate formula (3.33) for the line half-width $\Delta\omega^{\mathtt{P}}_{\mathtt{R}k}$ in the resonance region of the power absorption $P^{(0)}_{k}(\omega)$, scaled by J_1, is displayed varying the daming constant $\gamma_{\mathtt{R}k}$ of the phonon reservoir, scaled by J_1, from 0.5 to 5.5 for the cases of anisotropy energies $\hbar K$ given by $A=K/J_1=1.5,2.0,3.0,5.0$, and for the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1)=1.0$ and the spin-magnitude $S=5/2$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$.

Figure 20: The approximate formula (3.45) for the line half-width $\Delta\omega^{\mathtt{M}}_{\mathtt{R}k}$ in the resonance region of the magnetization-amplitudes $A^{\mathtt{M}(0)}_{k}(\omega)$, scaled by J_1, is displayed varying the daming constant $\gamma_{\mathtt{R}k}$ of the phonon reservoir, scaled by J_1, from 0.5 to 5.5 for the cases of anisotropy energies $\hbar K$ given by $A=K/J_1=1.5,2.0,3.0,5.0$, and for the temperature T given by $k_{\mathtt{B}}T/(\hbar J_1)=1.0$ and the spin-magnitude $S=5/2$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$.

becomes high as seen in Figs. 13 and 14, and also become large as the phonon reservoir is damped slowly or as the wave number k becomes small as seen in Figs. 15 and 16, and thus those effects produce effects that cannot be neglected for the high temperature, for the non-quickly damped reservoir or for the small wave number k. If the phonon reservoir is damped quickly, that is to say, the relaxation time τ_r of the spin system is much greater than the correlation time τ_c of the phonon reservoir, i.e., $\tau_r \gg \tau_c$, as being discussed in Ref. [25], one obtains the transverse susceptibility $\chi^{\mathrm{rv}}_{S^+_k S^-_k}(\omega)$ given by (3.46) without the interference thermal state $|D^{(2)}_{S^-_k}[\omega]\rangle$ in the transverse susceptibility $\chi_{S^+_k S^-_k}(\omega)$ [(3.6)] derived employing the TCLE method [25]. The susceptibility $\chi^{\mathrm{rv}}_{S^+_k S^-_k}(\omega)$ is derived employing the relaxation method [25] in the van Hove limit [39] or in the narrowing limit [40], and is valid only in the limit in which the phonon reservoir is damped quickly [25]. Since the transverse relaxation times of the anti-ferromagnetic spin system are equal to $(\Gamma'_{k\pm})^{-1}$ according to (A.57a) and (A.57b), where $\Gamma'_{k\pm}$ is given by (A.59a) or (4.8a), and the transverse correlation time of the phonon reservoir is equal to $(\gamma_{\mathtt{R}k})^{-1}$ according to (4.1a) or (4.1b), we have $(\Gamma'_{k\pm})^{-1} \gg (\gamma_{\mathtt{R}k})^{-1}$, i.e., $\Gamma'_{k\pm} \ll \gamma_{\mathtt{R}k}$, or (the transverse correlation time $(\gamma_{\mathtt{R}k})^{-1}=\tau^{\mathtt{T}}_c$ of the phonon reservoir) $\to 0$ in the van Hove limit [39] or in the narrowing limit [40]. In this limit, since the corresponding interference terms $X^{\alpha(\beta)}_{k1}(\omega)$ and $X^{\alpha(\beta)}_{k2}(\omega)$ vanish according to (C.3) − (C.6) as seen in Figs. 15 and 16, the transverse susceptibility becomes $\chi^{\mathrm{rv}}_{S^+_k S^-_k}(\omega)$ given by (3.46), and therefore one cannot discuss theoretically variations of the peak-heights and line half-widths in the resonance region of the power-absorption and magnetization-amplitude, because the peak-heights approach to ∞ and the line half-widths approach to 0 in that limit as seen in Figs. 17 − 20. The transverse magnetic susceptibility $\chi_{S^+_k S^-_k}(\omega)$ derived employing the second-order TCLE method is valid even if the phonon reservoir is damped slowly, in the region valid for the second-order perturbation approximation. Thus, the TCLE method is available for a spin system interacting with a non-quickly damped phonon-reservoir as well, and one can discuss theoretically variations of the peak-heights and line half-widths in the resonance region of the power-absorption and magnetization-amplitude derived employing the TCLE method, whereas one cannot discuss theoretically variations of the peak-heights and line half-widths employing the relaxation method [25] in the van Hove limit [39] or in the narrowing limit [40], in which the phonon reservoir is damped quickly [25].

We have analytically examined the power absorption and magnetization-amplitude in the resonance region of an anti-ferromagnetic spin system interacting with a phonon reservoir using the spin-wave method [4, 7], and have derived the approximate formulas of the resonance frequencies, peak-heights (heights of peak) and line half-widths in the lowest spin-wave approximation. We have numerically investigated an anti-ferromagnetic system of one-dimensional ininite spins in the region valid for the lowest spin-wave approximation, and have shown that the approximate formulas of the resonance frequencies, peak-heights and line half-widths in the resonance region, coincide well with the results investigated calculating numerically the analytic results of the power absorption and magnetization-amplitude, and satisfy “the narrowing condition” that as phonon reservoir is damped quickly, the peak-heights increase and the line half-widths decrease, and thus we have numerically verified the approximate formulas. The approximate formulas obtained for the resonance frequencies, peak-heights and line half-widths in the resonance region, may have to be verified for the various cases both experimentally and by the other theoretical method, e.g. the simulation method. We have also shown numerically that the energy transfer between the spin system and phonon reservoir at the "down" spin sites has few influence on the power absorptions and magnetization-amplitudes. We have besides investigated numerically the effects of the memory and initial correlation for the spin system and phonon reservoir, i.e., the interference effects (the effects of interference between the external driving field and the phonon reservoir), and have shown that those effects produce effects that cannot be neglected for the high temperature, for the non-quickly damped reservoir or for the small wave-number. Although the numerical investigation have been performed for an anti-ferromagnetic system of one-dimensional infinite spins, the analytic results obtained in the present paper are available for two- and three-dimensional spin systems as well.

Appendix

A NETFD for anti-ferromagnetic spin system

In this Appendix, we consider the anti-ferromagnetic spin system interacting with the phonon reservoir, which has been modeled in Section 2, and reformulate the non-equilibrium thermo-field dynamics (NETFD) for the spin-phonon interaction (2.21) taken to reflect the energy transfer between the spin system and phonon reservoir.

A.1 Basic formulation

We first provide the time-convolutionless (TCL) equation of motion for the anti-ferromagnetic spin system and phonon reservoir. We take the Hamiltonian $\mathcal{H}$ of the anti-ferromagnetic system and phonon reservoir under an external static magnetic-field, as

$$\mathcal{H} = \mathcal{H}_{\mathtt{S}} + \mathcal{H}_{\mathtt{R}} + \mathcal{H}_{\mathtt{SR}} = \mathcal{H}_0 + \mathcal{H}_{\mathtt{SR}}\,, \qquad (\mathcal{H}_0 = \mathcal{H}_{\mathtt{S}} + \mathcal{H}_{\mathtt{R}}), \tag{A.1}$$

and provide the basic requirements (axioms)

$$\hat{\mathcal{H}}\,|\rho_{\mathtt{TE}}\rangle = 0, \qquad \hat{\mathcal{H}}_{\mathtt{S}}\,|\rho_{\mathtt{S}}\rangle = 0, \qquad \hat{\mathcal{H}}_{\mathtt{R}}\,|\rho_{\mathtt{R}}\rangle = 0, \tag{A.2}$$

as in Ref. [28], where $\rho_{\mathtt{TE}}$ and $\rho_{\mathtt{S}}$ are the normalized, time-independent density operators given by

$$\rho_{\mathtt{TE}} = \exp(-\beta\,\mathcal{H})/\langle 1|\exp(-\beta\,\mathcal{H})\rangle = \exp(-\beta\,\mathcal{H})/\operatorname{Tr}\exp(-\beta\,\mathcal{H}), \tag{A.3}$$

$$\rho_{\mathtt{S}} = \exp(-\beta\,\mathcal{H}_{\mathtt{S}})/\langle 1_{\mathtt{S}}|\exp(-\beta\,\mathcal{H}_{\mathtt{S}})\rangle = \exp(-\beta\,\mathcal{H}_{\mathtt{S}})/\operatorname{tr}\exp(-\beta\,\mathcal{H}_{\mathtt{S}}), \tag{A.4}$$

which are the thermal equilibrium density operators at temperature $T=(k_{\mathtt{B}}\beta)^{-1}$, where $\operatorname{Tr}=\operatorname{tr}\operatorname{tr}_{\mathtt{R}}$. Here, $\hat{\mathcal{H}}$, $\hat{\mathcal{H}}_{\mathtt{S}}$ and $\hat{\mathcal{H}}_{\mathtt{R}}$ are the renormalized hat-Hamiltonians defined by, for example, $\hat{\mathcal{H}}=(\mathcal{H}-\tilde{\mathcal{H}}^{\dagger})/\hbar$ [25]. The spin deviation operators α_k, β_k, the phonon operators $R^a_{k\nu}$, $R^b_{k\nu}$ and their tilde conjugates satisfy the commutation relations

$$\big[\alpha_k\,,\,\alpha^{\dagger}_{k'}\big] = \big[\tilde{\alpha}_k\,,\,\tilde{\alpha}^{\dagger}_{k'}\big] = \big[\beta_k\,,\,\beta^{\dagger}_{k'}\big] = \big[\tilde{\beta}_k\,,\,\tilde{\beta}^{\dagger}_{k'}\big] = \delta_{kk'}\,, \tag{A.5}$$

$$\big[R^a_{k\nu}\,,\,R^{a\dagger}_{k'\nu'}\big] = \big[\tilde{R}^a_{k\nu}\,,\,\tilde{R}^{a\dagger}_{k'\nu'}\big] = \big[R^b_{k\nu}\,,\,R^{b\dagger}_{k'\nu'}\big] = \big[\tilde{R}^b_{k\nu}\,,\,\tilde{R}^{b\dagger}_{k'\nu'}\big] = \delta_{kk'}\,\delta_{\nu\nu'}\,, \tag{A.6}$$

while the other commutators vanish. As done in Refs. [21, 22, 28], we provide the basic requirements

$$\langle 1_{\mathtt{S}}|\alpha_k = \langle 1_{\mathtt{S}}|\tilde{\alpha}^{\dagger}_k\,, \qquad \langle 1_{\mathtt{S}}|\beta_k = \langle 1_{\mathtt{S}}|\tilde{\beta}^{\dagger}_k\,, \tag{A.7}$$

$$\langle 1_{\mathtt{R}}|R^a_{k\nu} = \langle 1_{\mathtt{R}}|\tilde{R}^{a\dagger}_{k\nu}\,, \qquad \langle 1_{\mathtt{R}}|R^b_{k\nu} = \langle 1_{\mathtt{R}}|\tilde{R}^{b\dagger}_{k\nu}\,, \tag{A.8}$$

and their tilde conjugates.

In the thermal-Liouville space of the spin system and phonon reservoir, the time-evolution of the thermal state $|\rho_{\mathtt{T}}(t)\rangle$ $[=\rho_{\mathtt{T}}(t)|1\rangle]$ for the density operator $\rho_{\mathtt{T}}(t)$ of the total system is given by the *Schrödinger equation* [26, 27, 28]

$$(d/dt)\,|\rho_{\mathtt{T}}(t)\rangle = -\,i\,\hat{\mathcal{H}}\,|\rho_{\mathtt{T}}(t)\rangle. \tag{A.9}$$

The spin system and phonon reservoir are assumed to be in the thermal state $|\rho_{\mathtt{T}}(0)\rangle$ at the initial time $t=0$ as an initial condition. In order to eliminate the irrelevant part associated with the phonon reservoir, we introduce the time-independent projection operators $\mathcal{P}$ and $\mathcal{Q}$ defined by [27]

$$\mathcal{P} = |\rho_{\mathtt{R}}\rangle\langle 1_{\mathtt{R}}| = \rho_{\mathtt{R}}\,|1_{\mathtt{R}}\rangle\langle 1_{\mathtt{R}}| \qquad \text{and} \qquad \mathcal{Q} = 1-\mathcal{P}. \tag{A.10}$$

Proceeding in the same way as in Ref. [47], the time-convolutionless (TCL) equation of motion for the reduced thermal state $|\rho(t)\rangle$ $[=\langle 1_{\mathtt{R}}|\rho_{\mathtt{T}}(t)\rangle]$ can be obtained as [29, 30]

$$(d/dt)\,|\rho(t)\rangle = -\,i\,\hat{\mathcal{H}}_{\mathtt{S}}\,|\rho(t)\rangle + C(t)\,|\rho(t)\rangle + |I(t)\rangle, \tag{A.11}$$

where the collision operator $C(t)$ and the thermal state $|I(t)\rangle$ are given by

$$C(t) = -\,i\,\langle 1_{\mathrm{R}}|\,\hat{\mathcal{H}}_{\mathrm{SR}}\,\{\Theta(t)-1\}\,|\rho_{\mathrm{R}}\rangle, \tag{A.12}$$
$$|I(t)\rangle = -\,i\,\langle 1_{\mathrm{R}}|\,\hat{\mathcal{H}}_{\mathrm{SR}}\,\Theta(t)\exp(-\,i\,\mathcal{Q}\hat{\mathcal{H}}\,\mathcal{Q}\,t\,)\,\mathcal{Q}\,|\rho_{\mathrm{T}}(0)\rangle, \tag{A.13}$$

with $\Theta(t)$ defined by

$$\Theta(t) = \Big\{1+i\int_0^t d\tau\exp(-\,i\,\mathcal{Q}\hat{\mathcal{H}}\,\mathcal{Q}\,\tau)\,\mathcal{Q}\hat{\mathcal{H}}\,\mathcal{P}\exp(i\,\hat{\mathcal{H}}\,\tau)\Big\}^{-1}. \tag{A.14}$$

Here, we have adopted the first order renormalization given by (2.19) – (2.21) for the free spin-wave Hamiltonian, the free spin-wave energies and the spin-phonon interaction. The thermal state $|I(t)\rangle$ depends on the initial condition of the spin system and phonon reservoir, and represents the effects of the initial correlation for the spin system and phonon reservoir.

We now consider the case that the spin system is interacting so weakly with the phonon reservoir that we can use the second-order approximation, and expand Eq. (A.11) up to the second order in powers of the spin-phonon interaction. When we assume the initial condition that the spin system and phonon reservoir are in the thermal equilibrium state at the initial time $t=0$, i.e., $|\rho_{\mathrm{T}}(0)\rangle=|\rho_{\mathrm{TE}}\rangle$, Eq. (A.11) reduces to

$$(d/dt)\,|\rho(t)\rangle = -\,i\,\hat{\mathcal{H}}_{\mathrm{S}}\,|\rho(t)\rangle + C^{(2)}(t)\,|\rho(t)\rangle + |I^{(2)}(t)\rangle, \tag{A.15}$$

where $C^{(2)}(t)$ and $|I^{(2)}(t)\rangle$ are given by [29, 30]

$$C^{(2)}(t) = -\int_0^t d\tau\,\big\langle 1_{\mathrm{R}}\big|\,\hat{\mathcal{H}}_{\mathrm{SR}}\exp(-\,i\,\hat{\mathcal{H}}_0\,\tau)\,\hat{\mathcal{H}}_{\mathrm{SR}}\exp(i\,\hat{\mathcal{H}}_0\,\tau)\,\big|\rho_{\mathrm{R}}\big\rangle, \tag{A.16}$$

$$\begin{aligned}|I^{(2)}(t)\rangle &= i\,\big\langle 1_{\mathrm{R}}\big|\,\hat{\mathcal{H}}_{\mathrm{SR}}\exp(-\,i\,\hat{\mathcal{H}}_0\,t)\int_0^\beta d\beta'\,\rho_{\mathrm{S}}\,\rho_{\mathrm{R}}\exp(\beta'\hbar\,\hat{\mathcal{H}}_0)\,\big|\mathcal{H}_{\mathrm{SR}}\big\rangle,\\ &= -\lim_{\mu\to+0}\int_t^\infty d\tau\,\big\langle 1_{\mathrm{R}}\big|\,\hat{\mathcal{H}}_{\mathrm{SR}}\exp(-\,i\,\hat{\mathcal{H}}_0\,\tau)\,\hat{\mathcal{H}}_{\mathrm{SR}}\,\rho_{\mathrm{S}}\,\rho_{\mathrm{R}}\,\big|1\big\rangle\,e^{-\,\mu\,\tau}.\end{aligned} \tag{A.17}$$

If the relaxation time τ_r of the spin system is much greater than the correlation time τ_c of the phonon reservoir, i.e., $\tau_r \gg \tau_c$, the thermal state $|I^{(2)}(t)\rangle$ becomes small negligibly [25, 29, 30, 48]. Thus, in the case that the relaxation time τ_r of the spin system is much larger than the correlation time τ_c of the phonon reservoir, i.e., $\tau_r \gg \tau_c$, which corresponds to the van Hove limit [39] or the narrowing limit [40], the phonon reservoir is damped quickly, and we have $C^{(2)}(t)=C^{(2)}(\infty)$ and $|I^{(2)}(t)\rangle=0$. In this Appendix, we consider such a case. Then, the reduced thermal state $|\rho(t)\rangle$ $[=\langle 1_{\mathrm{R}}|\rho_{\mathrm{T}}(t)\rangle\,]$ satisfies the following equation and initial condition

$$(d/dt)\,|\rho(t)\rangle = -\,i\,\hat{\mathcal{H}}_{\mathrm{S}}\,|\rho(t)\rangle + C^{(2)}\,|\rho(t)\rangle\ ; \qquad |\rho(0)\rangle = \langle 1_{\mathrm{R}}|\rho_{\mathrm{T}}(0)\rangle = \langle 1_{\mathrm{R}}|\rho_{\mathrm{TE}}\rangle, \tag{A.18}$$

for $\tau_r \gg \tau_c$, where the collision operator $C^{(2)}$ is defined by

$$C^{(2)} = C^{(2)}(\infty) = -\int_0^\infty d\tau\,\big\langle 1_{\mathrm{R}}\big|\,\hat{\mathcal{H}}_{\mathrm{SR}}\exp(-\,i\,\hat{\mathcal{H}}_0\,\tau)\,\hat{\mathcal{H}}_{\mathrm{SR}}\exp(i\,\hat{\mathcal{H}}_0\,\tau)\,\big|\rho_{\mathrm{R}}\big\rangle. \tag{A.19}$$

Equation (A.18) is can be formally solved as

$$|\rho(t)\rangle = \exp\{-\,i\,\hat{\mathcal{H}}_{\mathrm{S}}\,t + C^{(2)}\,t\}|\rho_0\rangle = U(t)\exp_{\leftarrow}\Big\{-\,i\int_0^t d\tau\,\hat{\mathcal{H}}_{\mathrm{S1}}(\tau)\Big\}|\rho_0\rangle, \qquad (\tau_r \gg \tau_c), \tag{A.20}$$

with $\rho_0=\rho(0)=\mathrm{tr}_{\mathrm{R}}\,\rho_{\mathrm{TE}}$, i.e., $|\rho_0\rangle=|\rho(0)\rangle=\langle 1_{\mathrm{R}}|\rho_{\mathrm{TE}}\rangle$. Here, we have divided the Hamiltonian $\mathcal{H}_{\mathrm{S}}$ of the spin system into the unperturbed part $\mathcal{H}_{\mathrm{S0}}$ and the perturbed part $\mathcal{H}_{\mathrm{S1}}$, i.e., $\mathcal{H}_{\mathrm{S}}=\mathcal{H}_{\mathrm{S0}}+\mathcal{H}_{\mathrm{S1}}$, and have defined

$$U(t) = \exp\{-\,(i\,\hat{\mathcal{H}}_{\mathrm{S0}} - C^{(2)})\,t\,\} = \exp\{-\,i\,(\hat{\mathcal{H}}_{\mathrm{S0}} + i\,C^{(2)})\,t\,\}, \tag{A.21}$$
$$\hat{\mathcal{H}}_{\mathrm{S1}}(t) = U^{-1}(t)\,\hat{\mathcal{H}}_{\mathrm{S1}}\,U(t), \qquad [\,\hat{\mathcal{H}}_{\mathrm{S1}} = (\mathcal{H}_{\mathrm{S1}} - \tilde{\mathcal{H}}_{\mathrm{S1}}^{\dagger})/\hbar\,]. \tag{A.22}$$

Then, the expectation value of a physical quantity A of the spin system can be described as

$$\langle 1|A|\rho_{\mathrm{T}}(t)\rangle = \langle 1_{\mathrm{S}}|A|\rho(t)\rangle = \langle 1_{\mathrm{S}}|A\,U(t)\exp_{\leftarrow}\Big\{-\,i\int_0^t d\tau\,\hat{\mathcal{H}}_{\mathrm{S1}}(\tau)\Big\}\,|\rho_0\rangle. \tag{A.23}$$

This expression is convenient for the expansion in powers of the spin-wave interaction $\mathcal{H}_{\mathrm{S1}}$.

A.2 Collision operator and thermal-state conditions

By substituting (2.21) into (A.19) and by using the basic requirements (A.8) and their tilde conjugates, we can derive the concrete expression of the collision operator $C^{(2)}$ given by (A.19), as

$$
\begin{aligned}
C^{(2)} = \frac{-S}{2} \sum_k \{ & (\phi_k^{+-}(\epsilon_k^+) + \phi_k^{-+}(-\epsilon_k^+))\{(\alpha_k - \tilde{\alpha}_k^\dagger)\, \alpha_k^\dagger \cosh 2\theta_k - (\beta_k^\dagger - \tilde{\beta}_k)\, \alpha_k^\dagger \sinh 2\theta_k\} \\
& - (\phi_k^{-+}(\epsilon_k^+)^* + \phi_k^{+-}(-\epsilon_k^+)^*)\{(\alpha_k - \tilde{\alpha}_k^\dagger)\, \tilde{\alpha}_k \cosh 2\theta_k - (\beta_k^\dagger - \tilde{\beta}_k)\, \tilde{\alpha}_k \sinh 2\theta_k\} \\
& + (\phi_k^{-+}(\epsilon_k^+) + \phi_k^{+-}(-\epsilon_k^+))\{(\alpha_k^\dagger - \tilde{\alpha}_k)\, \alpha_k \cosh 2\theta_k - (\beta_k - \tilde{\beta}_k^\dagger)\, \alpha_k \sinh 2\theta_k\} \\
& - (\phi_k^{+-}(\epsilon_k^+)^* + \phi_k^{-+}(-\epsilon_k^+)^*)\{(\alpha_k^\dagger - \tilde{\alpha}_k)\, \tilde{\alpha}_k^\dagger \cosh 2\theta_k - (\beta_k - \tilde{\beta}_k^\dagger)\, \tilde{\alpha}_k^\dagger \sinh 2\theta_k\} \\
& + (\phi_k^{+-}(-\epsilon_k^-) + \phi_k^{-+}(\epsilon_k^-))\{(\beta_k^\dagger - \tilde{\beta}_k)\, \beta_k \cosh 2\theta_k - (\alpha_k - \tilde{\alpha}_k^\dagger)\, \beta_k \sinh 2\theta_k\} \\
& - (\phi_k^{-+}(-\epsilon_k^-)^* + \phi_k^{+-}(\epsilon_k^-)^*)\{(\beta_k^\dagger - \tilde{\beta}_k)\, \tilde{\beta}_k^\dagger \cosh 2\theta_k - (\alpha_k - \tilde{\alpha}_k^\dagger)\, \tilde{\beta}_k^\dagger \sinh 2\theta_k\} \\
& + (\phi_k^{-+}(-\epsilon_k^-) + \phi_k^{+-}(\epsilon_k^-))\{(\beta_k - \tilde{\beta}_k^\dagger)\, \beta_k^\dagger \cosh 2\theta_k - (\alpha_k^\dagger - \tilde{\alpha}_k)\, \beta_k^\dagger \sinh 2\theta_k\} \\
& - (\phi_k^{+-}(-\epsilon_k^-)^* + \phi_k^{-+}(\epsilon_k^-)^*)\{(\beta_k - \tilde{\beta}_k^\dagger)\, \tilde{\beta}_k \cosh 2\theta_k - (\alpha_k^\dagger - \tilde{\alpha}_k)\, \tilde{\beta}_k \sinh 2\theta_k\}\} \\
- \frac{S}{2} \sum_k \{ & (\phi_k^{+-}(\epsilon_k^+) - \phi_k^{-+}(-\epsilon_k^+))(\alpha_k - \tilde{\alpha}_k^\dagger)\alpha_k^\dagger - (\phi_k^{-+}(\epsilon_k^+)^* - \phi_k^{+-}(-\epsilon_k^+)^*)(\alpha_k - \tilde{\alpha}_k^\dagger)\tilde{\alpha}_k \\
& + (\phi_k^{-+}(\epsilon_k^+) - \phi_k^{+-}(-\epsilon_k^+))(\alpha_k^\dagger - \tilde{\alpha}_k)\alpha_k - (\phi_k^{+-}(\epsilon_k^+)^* - \phi_k^{-+}(-\epsilon_k^+)^*)(\alpha_k^\dagger - \tilde{\alpha}_k)\tilde{\alpha}_k^\dagger \\
& - (\phi_k^{+-}(-\epsilon_k^-) - \phi_k^{-+}(\epsilon_k^-))(\beta_k^\dagger - \tilde{\beta}_k)\beta_k + (\phi_k^{-+}(-\epsilon_k^-)^* - \phi_k^{+-}(\epsilon_k^-)^*)(\beta_k^\dagger - \tilde{\beta}_k)\tilde{\beta}_k^\dagger \\
& - (\phi_k^{-+}(-\epsilon_k^-) - \phi_k^{+-}(\epsilon_k^-))(\beta_k - \tilde{\beta}_k^\dagger)\beta_k^\dagger + (\phi_k^{+-}(-\epsilon_k^-)^* - \phi_k^{-+}(\epsilon_k^-)^*)(\beta_k - \tilde{\beta}_k^\dagger)\tilde{\beta}_k\} \\
- \frac{1}{2} \sum_k \{ & \{(\alpha_k^\dagger\alpha_k - \tilde{\alpha}_k^\dagger\tilde{\alpha}_k + \beta_k^\dagger\beta_k - \tilde{\beta}_k^\dagger\tilde{\beta}_k) \cosh 2\theta_k - (\alpha_k\beta_k + \alpha_k^\dagger\beta_k^\dagger - \tilde{\alpha}_k^\dagger\tilde{\beta}_k^\dagger - \tilde{\alpha}_k\tilde{\beta}_k) \sinh 2\theta_k\} \\
& \quad \times \{(\alpha_k^\dagger\alpha_k - \tilde{\alpha}_k^\dagger\tilde{\alpha}_k + \beta_k^\dagger\beta_k - \tilde{\beta}_k^\dagger\tilde{\beta}_k) \cosh 2\theta_k\, \phi_k^{zz}(0) \\
& \qquad - ((\alpha_k\beta_k - \tilde{\alpha}_k^\dagger\tilde{\beta}_k^\dagger)\, \phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-) + (\alpha_k^\dagger\beta_k^\dagger - \tilde{\alpha}_k\tilde{\beta}_k)\, \phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-)^*) \sinh 2\theta_k\} \\
& + \{\alpha_k^\dagger\alpha_k - \tilde{\alpha}_k^\dagger\tilde{\alpha}_k - (\beta_k^\dagger\beta_k - \tilde{\beta}_k^\dagger\tilde{\beta}_k)\}\{\alpha_k^\dagger\alpha_k - \tilde{\alpha}_k^\dagger\tilde{\alpha}_k - (\beta_k^\dagger\beta_k - \tilde{\beta}_k^\dagger\tilde{\beta}_k)\}\, \phi_k^{zz}(0)\},
\end{aligned}
\tag{A.24}
$$

where $\phi_k^{+-}(\epsilon)$, $\phi_k^{-+}(\epsilon)$ and $\phi_k^{zz}(\epsilon)$ are given by

$$\phi_k^{+-}(\epsilon) = \frac{1}{2} \sum_\nu |\, g_{1\nu}|^2 \int_0^\infty d\tau \, \langle 1_{\mathtt{R}}|\, R_{k\nu}^\dagger(\tau)\, R_{k\nu} \,|\rho_{\mathtt{R}}\rangle \exp(-\,i\,\epsilon\,\tau), \tag{A.25a}$$

$$\phi_k^{-+}(\epsilon) = \frac{1}{2} \sum_\nu |\, g_{1\nu}|^2 \int_0^\infty d\tau \, \langle 1_{\mathtt{R}}|\, R_{k\nu}(\tau)\, R_{k\nu}^\dagger \,|\rho_{\mathtt{R}}\rangle \exp(i\,\epsilon\,\tau), \tag{A.25b}$$

$$\phi_k^{zz}(\epsilon) = \sum_\nu g_{2\nu}^2 \int_0^\infty d\tau \big\langle 1_{\mathtt{R}}\big|\Delta(R_{k\nu}^\dagger(\tau)R_{k\nu}(\tau))\Delta(R_{k\nu}^\dagger R_{k\nu})\big|\rho_{\mathtt{R}}\big\rangle \exp(i\,\epsilon\,\tau). \tag{A.25c}$$

In the derivation of the above form for the collision operator $C^{(2)}$, we have ignored the correlation between the first term and second term in the spin-phonon interaction (2.21), and have neglected the spin-wave interaction $\mathcal{H}_{\mathtt{S}1}$ in the Hamiltonian $\mathcal{H}_{\mathtt{S}}$ of the spin system. The basic requirements (A.7) and their tilde conjugates lead to

$$\langle 1_{\mathtt{S}}|\, C^{(2)} = 0, \qquad\qquad \langle 1_{\mathtt{S}}|\, U(t) = \langle 1_{\mathtt{S}}|\, U^{-1}(t) = \langle 1_{\mathtt{S}}|, \tag{A.26}$$

for $U(t)$ defined by (A.21). The evolution operator $U(t)$ is non-unitary in general, i.e., $U^\dagger(t) \neq U^{-1}(t)$, because the collision operator $C^{(2)}$ is non-Hermitian though $\hat{\mathcal{H}}_{\mathtt{S}0}$ $[= (\mathcal{H}_{\mathtt{S}0} - \tilde{\mathcal{H}}_{\mathtt{S}0}^\dagger)/\hbar]$ is Hermitian. Therefore, for $t \neq 0$, we have $(U^{-1}(t)\alpha_k U(t))^\dagger \neq U^{-1}(t)\alpha_k^\dagger U(t)$ and $(U^{-1}(t)\tilde{\alpha}_k U(t))^\dagger \neq U^{-1}(t)\tilde{\alpha}_k^\dagger U(t)$ and so for β, $\tilde{\beta}$. Considering this, as done in Refs. [21, 22, 27, 28], we define the Heisenberg operators as

$$\alpha_k(t) = U^{-1}(t)\, \alpha_k\, U(t), \qquad\qquad \alpha_k^{\dagger\dagger}(t) = U^{-1}(t)\, \alpha_k^\dagger\, U(t), \tag{A.27a}$$

$$\beta_k(t) = U^{-1}(t)\, \beta_k\, U(t), \qquad\qquad \beta_k^{\dagger\dagger}(t) = U^{-1}(t)\, \beta_k^\dagger\, U(t), \tag{A.27b}$$

and their tilde conjugates, which satisfy the canonical commutation relations

$$[\, \alpha_k(t)\, ,\, \alpha_{k'}^{\dagger\dagger}(t)\,] = [\, \tilde{\alpha}_k(t)\, ,\, \tilde{\alpha}_{k'}^{\dagger\dagger}(t)\,] = [\, \beta_k(t)\, ,\, \beta_{k'}^{\dagger\dagger}(t)\,] = [\, \tilde{\beta}_k(t)\, ,\, \tilde{\beta}_{k'}^{\dagger\dagger}(t)\,] = \delta_{k\,k'}\, , \tag{A.28}$$

while the other commutators vanish. According to the axioms (A.7), (A.26) and their tilde conjugates, we have

$$\langle 1_{\mathtt{S}}|\alpha_k(t) = \langle 1_{\mathtt{S}}|\tilde{\alpha}_k^{\dagger\dagger}(t), \qquad\qquad \langle 1_{\mathtt{S}}|\beta_k(t) = \langle 1_{\mathtt{S}}|\tilde{\beta}_k^{\dagger\dagger}(t), \tag{A.29}$$

and their tilde conjugates, which are the thermal-state conditions at time t for the bra-vector $\langle 1_{\mathtt{S}}|$ of the spin system. By proceeding as in Refs. [24, 28], the thermal-state conditions at time t for the ket-vector $|\rho_0\rangle$ $[=\rho_0|1_{\mathtt{S}}\rangle = \langle 1_{\mathtt{R}}|\rho_{\mathtt{TE}}\rangle]$ of the spin system, can be obtained as

$$\alpha_k(t)\,|\rho_0\rangle = h_k^\alpha(t)\,\tilde{\alpha}_k^{\dagger\dagger}(t)\,|\rho_0\rangle, \qquad \beta_k(t)\,|\rho_0\rangle = h_k^\beta(t)\,\tilde{\beta}_k^{\dagger\dagger}(t)\,|\rho_0\rangle, \tag{A.30}$$

and their tilde conjugates, where the c-number functions $h_k^\alpha(t)$ and $h_k^\beta(t)$ are given by

$$h_k^\alpha(t) = n_k^\alpha(t)\{1+n_k^\alpha(t)\}^{-1}\,; \qquad h_k^\beta(t) = n_k^\beta(t)\{1+n_k^\beta(t)\}^{-1}, \tag{A.31}$$

with the quantities $n_k^\alpha(t)$ and $n_k^\beta(t)$ defined by

$$n_k^\alpha(t) = \langle 1_{\mathtt{S}}|\alpha_k^{\dagger\dagger}(t)\alpha_k(t)|\rho_0\rangle, \qquad n_k^\beta(t) = \langle 1_{\mathtt{S}}|\beta_k^{\dagger\dagger}(t)\beta_k(t)|\rho_0\rangle. \tag{A.32}$$

Here, the bra-vector $\langle 1_{\mathtt{S}}|$ and ket-vector $|\rho_0\rangle$ are normalized, i.e., $\langle 1_{\mathtt{S}}|\rho_0\rangle = \mathrm{tr}\,\rho_0 = 1$, and ρ_0 is given by $\rho_0 = \mathrm{tr}_{\mathtt{R}}\,\rho_{\mathtt{TE}}$.

We now introduce the annihilation and creation quasi-particle operators defined by [21, 22]

$$\lambda_k(t) = Z_k^\alpha(t)^{1/2}\,\{\alpha_k(t) - h_k^\alpha(t)\,\tilde{\alpha}_k^{\dagger\dagger}(t)\}, \qquad \lambda_k^\ddagger(t) = Z_k^\alpha(t)^{1/2}\,\{\alpha_k^{\dagger\dagger}(t) - \tilde{\alpha}_k(t)\}, \tag{A.33a}$$

$$\xi_k(t) = Z_k^\beta(t)^{1/2}\,\{\beta_k(t) - h_k^\beta(t)\,\tilde{\beta}_k^{\dagger\dagger}(t)\}, \qquad \xi_k^\ddagger(t) = Z_k^\beta(t)^{1/2}\,\{\beta_k^{\dagger\dagger}(t) - \tilde{\beta}_k(t)\}, \tag{A.33b}$$

and their tilde conjugates, where the normalization factor $Z_k^\alpha(t)$ and $Z_k^\beta(t)$ are given by

$$Z_k^\alpha(t) = \{1-h_k^\alpha(t)\}^{-1} = 1 + n_k^\alpha(t), \qquad h_k^\alpha(t) = 1 - Z_k^\alpha(t)^{-1}, \tag{A.34a}$$

$$Z_k^\beta(t) = \{1-h_k^\beta(t)\}^{-1} = 1 + n_k^\beta(t), \qquad h_k^\beta(t) = 1 - Z_k^\beta(t)^{-1}. \tag{A.34b}$$

These lead to the canonical commutation relations of the quasi-particle operators:

$$[\,\lambda_k(t)\,,\,\lambda_{k'}^\ddagger(t)\,] = [\,\tilde{\lambda}_k(t)\,,\,\tilde{\lambda}_{k'}^\ddagger(t)\,] = [\,\xi_k(t)\,,\,\xi_{k'}^\ddagger(t)\,] = [\,\tilde{\xi}_k(t)\,,\,\tilde{\xi}_{k'}^\ddagger(t)\,] = \delta_{k\,k'}, \tag{A.35}$$

while the other commutators vanish. The thermal state conditions (A.29) and (A.30) and their tilde conjugates give

$$\langle 1_{\mathtt{S}}|\lambda_k^\ddagger(t) = 0, \qquad \langle 1_{\mathtt{S}}|\,\xi_k^\ddagger(t) = 0\,; \qquad \lambda_k(t)|\rho_0\rangle = 0, \qquad \xi_k(t)|\rho_0\rangle = 0, \tag{A.36}$$

and their tilde conjugates. According to Eqs. (A.36) and their tilde conjugates, $\langle 1_{\mathtt{S}}|$ and $|\rho_0\rangle$ are, respectively, called *the thermal vacuum bra-vector* and *the thermal vacuum ket-vector* for the spin system [27, 28]. Performing the inverse transformation of (A.33a), (A.33b), and their tilde conjugates, we have

$$\alpha_k(t) = Z_k^\alpha(t)^{1/2}\,\{\lambda_k(t) + h_k^\alpha(t)\,\tilde{\lambda}_k^\ddagger(t)\}, \qquad \alpha_k^{\dagger\dagger}(t) = Z_k^\alpha(t)^{1/2}\,\{\lambda_k^\ddagger(t) + \tilde{\lambda}_k(t)\}, \tag{A.37a}$$

$$\beta_k(t) = Z_k^\beta(t)^{1/2}\,\{\xi_k(t) + h_k^\beta(t)\,\tilde{\xi}_k^\ddagger(t)\}, \qquad \beta_k^{\dagger\dagger}(t) = Z_k^\beta(t)^{1/2}\,\{\xi_k^\ddagger(t) + \tilde{\xi}_k(t)\}, \tag{A.37b}$$

and their tilde conjugates. The free spin-wave hat-Hamiltonian $\hat{\mathcal{H}}_{\mathtt{S}0}$ takes the diagonal forms

$$\hat{\mathcal{H}}_{\mathtt{S}0} = \sum_k \{\epsilon_k^+\,(\alpha_k^\dagger\,\alpha_k - \tilde{\alpha}_k^\dagger\,\tilde{\alpha}_k) + \epsilon_k^-\,(\beta_k^\dagger\,\beta_k - \tilde{\beta}_k^\dagger\,\tilde{\beta}_k)\} = \sum_k \{\epsilon_k^+\,(\lambda_k^\ddagger\,\lambda_k - \tilde{\lambda}_k^\ddagger\,\tilde{\lambda}_k) + \epsilon_k^-\,(\xi_k^\ddagger\,\xi_k - \tilde{\xi}_k^\ddagger\,\tilde{\xi}_k)\}, \tag{A.38}$$

with $\lambda_k = \lambda_k(0)$, $\lambda_k^\ddagger = \lambda_k^\ddagger(0)$, $\xi_k = \xi_k(0)$ and $\xi_k^\ddagger = \xi_k^\ddagger(0)$.

A.3 Forms of the quasi-particle operators

We next derive the forms of the quasi-particle operators. The equations of motion for $n_k^\alpha(t)$ and $n_k^\beta(t)$ defined by (A.32) can be obtained, by using the thermal-state conditions (A.29) and (A.30), as

$$\frac{d}{dt}\,n_k^\alpha(t) = \langle 1_{\mathtt{S}}|\Big(\frac{d}{dt}\,\alpha_k^{\dagger\dagger}(t)\,\alpha_k(t)\ \ |\rho_0\rangle = \langle 1_{\mathtt{S}}|\,U^{-1}(t)\,[\,i\,\hat{\mathcal{H}}_{\mathtt{S}0} - C^{(2)},\,\alpha_k^\dagger\,\alpha_k\,]\,U(t)\,|\rho_0\rangle, \tag{A.39a}$$

$$\begin{aligned}
&= -\,(S/2)\,\{\{(\phi_k^{-+}(\epsilon_k^+) - \phi_k^{+-}(\epsilon_k^+)^*)^* + (\phi_k^{-+}(\epsilon_k^+) - \phi_k^{+-}(\epsilon_k^+)^*)\} \\
&\qquad + \{(\phi_k^{+-}(-\epsilon_k^+) - \phi_k^{-+}(-\epsilon_k^+)^*)^* + (\phi_k^{+-}(-\epsilon_k^+) - \phi_k^{-+}(-\epsilon_k^+)^*)\}\}\cosh 2\theta_k\,n_k^\alpha(t) \\
&\quad + (S/2)\,\{(\phi_k^{+-}(\epsilon_k^+) + \phi_k^{+-}(\epsilon_k^+)^*) + S_2(\phi_k^{-+}(-\epsilon_k^+) + \phi_k^{-+}(-\epsilon_k^+)^*)\}\cosh 2\theta_k \\
&\quad - (S/2)\,\{\{(\phi_k^{-+}(\epsilon_k^+) - \phi_k^{+-}(\epsilon_k^+)^*)^* + (\phi_k^{-+}(\epsilon_k^+) - \phi_k^{+-}(\epsilon_k^+)^*)\} \\
&\qquad - \{(\phi_k^{+-}(-\epsilon_k^+) - \phi_k^{-+}(-\epsilon_k^+)^*)^* + (\phi_k^{+-}(-\epsilon_k^+) - \phi_k^{-+}(-\epsilon_k^+)^*)\}\}\,n_k^\alpha(t) \\
&\quad + (S/2)\,\{(\phi_k^{+-}(\epsilon_k^+) + \phi_k^{+-}(\epsilon_k^+)^*) - (\phi_k^{-+}(-\epsilon_k^+) + \phi_k^{-+}(-\epsilon_k^+)^*)\} \\
&\quad + (1/2)\{\phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-) + \phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-)^*\}\sinh^2 2\theta_k\,(n_k^\alpha(t) + n_k^\beta(t) + 1),
\end{aligned} \tag{A.39b}$$

$$\begin{aligned}
&= -\,\{S\,(\Phi_k^+(\epsilon_k^+)' + \Phi_k^-(\epsilon_k^+)')\cosh 2\theta_k + S\,(\Phi_k^+(\epsilon_k^+)' - \Phi_k^-(\epsilon_k^+)') - \Psi_k'\sinh^2 2\theta_k\}\,n_k^\alpha(t) \\
&\quad + \{S\,(\Phi_k^+(\epsilon_k^+)' + \Phi_k^-(\epsilon_k^+)')\cosh 2\theta_k + S\,(\Phi_k^+(\epsilon_k^+)' - \Phi_k^-(\epsilon_k^+)')\}\,\bar{n}(\epsilon_k^+) \\
&\quad + \Psi_k'\sinh^2 2\theta_k\,n_k^\beta(t) + \Psi_k'\sinh^2 2\theta_k,
\end{aligned} \tag{A.39c}$$

$$\frac{d}{dt}\, n_k^{\beta}(t) = \langle 1_{\mathtt{S}} | \Big(\frac{d}{dt}\, \beta_k^{\dagger\dagger}(t)\, \beta_k(t) \quad |\rho_0\rangle = \langle 1_{\mathtt{S}} |\, U^{-1}(t)\, [\, i\, \hat{\mathcal{H}}_{\mathtt{S}0} - C^{(2)},\, \beta_k^{\dagger}\, \beta_k\,]\, U(t)\, |\rho_0\rangle, \tag{A.40a}$$

$$\begin{aligned}
&= -\,(S/2)\, \{\{(\phi_k^{+-}(-\epsilon_k^-) - \phi_k^{-+}(-\epsilon_k^-)^*) + (\phi_k^{+-}(-\epsilon_k^-) - \phi_k^{-+}(-\epsilon_k^-)^*)^*\} \\
&\qquad + \{(\phi_k^{-+}(\epsilon_k^-) - \phi_k^{+-}(\epsilon_k^-)^*) + (\phi_k^{-+}(\epsilon_k^-) - \phi_k^{+-}(\epsilon_k^-)^*)^*\}\} \cosh 2\theta_k\, n_k^{\beta}(t) \\
&\quad + (S/2)\, \{(\phi_k^{-+}(-\epsilon_k^-) + \phi_k^{-+}(-\epsilon_k^-)^*) + (\phi_k^{+-}(\epsilon_k^-) + \phi_k^{+-}(\epsilon_k^-)^*)\} \cosh 2\theta_k \\
&\quad + (S/2)\, \{\{(\phi_k^{+-}(-\epsilon_k^-) - \phi_k^{-+}(-\epsilon_k^-)^*) + (\phi_k^{+-}(-\epsilon_k^-) - \phi_k^{-+}(-\epsilon_k^-)^*)^*\} \\
&\qquad - \{(\phi_k^{-+}(\epsilon_k^-) - (\phi_k^{+-}(\epsilon_k^-)^*) + (\phi_k^{-+}(\epsilon_k^-) - \phi_k^{+-}(\epsilon_k^-)^*)^*\}\}\, n_k^{\beta}(t) \\
&\quad - (S/2)\, \{(\phi_k^{-+}(-\epsilon_k^-) + \phi_k^{-+}(-\epsilon_k^-)^*) - (\phi_k^{+-}(\epsilon_k^-) + \phi_k^{+-}(\epsilon_k^-)^*)\} \\
&\quad + (1/2) \{\phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-) + \phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-)^*\} \sinh^2 2\theta_k\, (n_k^{\alpha}(t) + n_k^{\beta}(t) + 1),
\end{aligned} \tag{A.40b}$$

$$\begin{aligned}
&= -\, \{S\, (\Phi_k^-(\epsilon_k^-)' + \Phi_k^+(\epsilon_k^-)') \cosh 2\theta_k - S\, (\Phi_k^-(\epsilon_k^-)' - \Phi_k^+(\epsilon_k^-)') - \Psi_k' \sinh^2 2\theta_k\}\, n_k^{\beta}(t) \\
&\quad + \{S\, (\Phi_k^-(\epsilon_k^-)' + \Phi_k^+(\epsilon_k^-)') \cosh 2\theta_k - S\, (\Phi_k^-(\epsilon_k^-)' - \Phi_k^+(\epsilon_k^-)')\}\, \bar{n}(\epsilon_k^-) \\
&\quad + \Psi_k' \sinh^2 2\theta_k\, n_k^{\alpha}(t) + \Psi_k' \sinh^2 2\theta_k,
\end{aligned} \tag{A.40c}$$

with $\bar{n}(\epsilon)$ defined by

$$\bar{n}(\epsilon) = \{\exp(\beta\, \hbar\, \epsilon) - 1\}^{-1} = \{\exp(\hbar\, \epsilon/(k_{\mathtt{B}} T)) - 1\}^{-1}, \tag{A.41}$$

where $\Phi_k^{\pm}(\epsilon)'$ and Ψ_k' are the real parts of $\Phi_k^{\pm}(\epsilon)$ $[\, = \Phi_k^{\pm}(\epsilon)' + i\, \Phi_k^{\pm}(\epsilon)''\,]$ and Ψ_k $[\, = \Psi_k' + i\, \Psi_k''\,]$, which are defined by

$$\Phi_k^+(\epsilon) = \phi_k^{-+}(\epsilon) - \phi_k^{+-}(\epsilon)^* = \frac{1}{2} \int_0^{\infty} d\tau \sum_{\nu} |g_{1\nu}|^2 \langle 1_{\mathtt{R}} | [R_{k\nu}(\tau), R_{k\nu}^{\dagger}] | \rho_{\mathtt{R}} \rangle \exp(i\, \epsilon\, \tau), \tag{A.42}$$

$$\Phi_k^-(\epsilon) = \phi_k^{+-}(-\,\epsilon) - \phi_k^{-+}(-\,\epsilon)^* = \frac{1}{2} \int_0^{\infty} d\tau \sum_{\nu} |g_{1\nu}|^2 \langle 1_{\mathtt{R}} | [R_{k\nu}^{\dagger}(\tau), R_{k\nu}] | \rho_{\mathtt{R}} \rangle \exp(i\, \epsilon\, \tau), \tag{A.43}$$

$$\Psi_k = \phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-) = \int_0^{\infty} d\tau \sum_{\nu} g_{2\nu}^2 \langle 1_{\mathtt{R}} | \Delta(R_{k\nu}^{\dagger}(\tau) R_{k\nu}(\tau)) \Delta(R_{k\nu}^{\dagger} R_{k\nu}) | \rho_{\mathtt{R}} \rangle \exp\{i\, (\epsilon_k^+ + \epsilon_k^-)\, \tau\}. \tag{A.44}$$

In the derivations of Eqs. (A.39c) and (A.40c), we have used the relations [21]

$$\phi_k^{+-}(\epsilon) + \phi_k^{+-}(\epsilon)^* = 2\, \bar{n}(\epsilon)\, \Phi_k^+(\epsilon)', \qquad \phi_k^{-+}(-\epsilon) + \phi_k^{-+}(-\epsilon)^* = 2\, \bar{n}(\epsilon)\, \Phi_k^-(\epsilon)', \tag{A.45}$$

which were derived in Appendix A of Ref. [21]. According to the assumption that the phonon correlation function (2.24c) is real, we have $\Psi_k' = (\phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-) + \phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-)^*)/2 = (\phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-) + \phi_k^{zz}(-\,\epsilon_k^+ - \epsilon_k^-))/2$. The solutions of Eqs. (A.39c) and (A.40c) can be written as

$$\begin{aligned}
n_k^{\alpha}(t) = \int_0^t d\tau\, \big\{ &\{S(\Phi_k^+(\epsilon_k^+)' + \Phi_k^-(\epsilon_k^+)') \cosh 2\theta_k + S(\Phi_k^+(\epsilon_k^+)' - \Phi_k^-(\epsilon_k^+)')\}\, \bar{n}(\epsilon_k^+) \\
&+ \Psi_k' \sinh^2 2\theta_k\, n_k^{\beta}(\tau) + \Psi_k' \sinh^2 2\theta_k \big\} \exp\{-\,\Gamma_{k+}^{\mathtt{L}} \cdot (t - \tau)\} + n_k^{\alpha}(0) \exp(-\,\Gamma_{k+}^{\mathtt{L}}\, t),
\end{aligned} \tag{A.46a}$$

$$\begin{aligned}
n_k^{\beta}(t) = \int_0^t d\tau\, \big\{ &\{S(\Phi_k^-(\epsilon_k^-)' + \Phi_k^+(\epsilon_k^-)') \cosh 2\theta_k - S(\Phi_k^-(\epsilon_k^-)' - \Phi_k^+(\epsilon_k^-)')\}\, \bar{n}(\epsilon_k^-) \\
&+ \Psi_k' \sinh^2 2\theta_k\, n_k^{\alpha}(\tau) + \Psi_k' \sinh^2 2\theta_k \big\} \exp\{-\,\Gamma_{k-}^{\mathtt{L}} \cdot (t - \tau)\} + n_k^{\beta}(0) \exp(-\,\Gamma_{k-}^{\mathtt{L}}\, t),
\end{aligned} \tag{A.46b}$$

with $n_k^{\alpha}(0) = \langle 1_{\mathtt{S}} | \alpha_k^{\dagger} \alpha_k | \rho_0 \rangle$ and $n_k^{\beta}(0) = \langle 1_{\mathtt{S}} | \beta_k^{\dagger} \beta_k | \rho_0 \rangle$, where we have put for brevity as

$$\Gamma_{k\pm}^{\mathtt{L}} = S\, (\Phi_k^{\pm}(\epsilon_k^{\pm})' + \Phi_k^{\mp}(\epsilon_k^{\pm})') \cosh 2\theta_k \pm S\, (\Phi_k^{\pm}(\epsilon_k^{\pm})' - \Phi_k^{\mp}(\epsilon_k^{\pm})') - \Psi_k' \sinh^2 2\theta_k. \tag{A.47}$$

By substituting each of the above forms for $n_k^{\alpha}(t)$ and $n_k^{\beta}(t)$ into the other, we obtain the approximate solutions as

$$\begin{aligned}
n_k^{\alpha}(t) = n_k^{\alpha}(0) \exp(-\,\Gamma_{k+}^{\mathtt{L}}\, t) &+ \Psi_k' \sinh^2 2\theta_k\, n_k^{\beta}(0)\, \frac{\exp(-\,\Gamma_{k-}^{\mathtt{L}}\, t) - \exp(-\,\Gamma_{k+}^{\mathtt{L}}\, t)}{\Gamma_{k+}^{\mathtt{L}} - \Gamma_{k-}^{\mathtt{L}}} \\
&+ \big\{ \{S\, (\Phi_k^+(\epsilon_k^+)' + \Phi_k^-(\epsilon_k^+)') \cosh 2\theta_k + S\, (\Phi_k^+(\epsilon_k^+)' - \Phi_k^-(\epsilon_k^+)')\}\, \bar{n}(\epsilon_k^+) \\
&\qquad + \Psi_k' \sinh^2 2\theta_k \big\} \{1 - \exp(-\,\Gamma_{k+}^{\mathtt{L}}\, t)\} / \, \Gamma_{k+}^{\mathtt{L}} + O(g^4),
\end{aligned} \tag{A.48a}$$

$$\begin{aligned}
n_k^{\beta}(t) = n_k^{\beta}(0) \exp(-\,\Gamma_{k-}^{\mathtt{L}}\, t) &+ \Psi_k' \sinh^2 2\theta_k\, n_k^{\alpha}(0)\, \frac{\exp(-\,\Gamma_{k+}^{\mathtt{L}}\, t) - \exp(-\,\Gamma_{k-}^{\mathtt{L}}\, t)}{\Gamma_{k-}^{\mathtt{L}} - \Gamma_{k+}^{\mathtt{L}}} \\
&+ \big\{ \{S\, (\Phi_k^-(\epsilon_k^-)' + \Phi_k^+(\epsilon_k^-)') \cosh 2\theta_k - S\, (\Phi_k^-(\epsilon_k^-)' - \Phi_k^+(\epsilon_k^-)')\}\, \bar{n}(\epsilon_k^-) \\
&\qquad + \Psi_k' \sinh^2 2\theta_k \big\} \{1 - \exp(-\,\Gamma_{k-}^{\mathtt{L}}\, t)\} / \, \Gamma_{k-}^{\mathtt{L}} + O(g^4),
\end{aligned} \tag{A.48b}$$

where $O(g^4)$ denotes the fourth-order parts in powers of the spin-phonon interaction. Owing to stability of the antiferromagnetic spin system, we assume that $\Gamma^{\mathtt{L}}_{k\pm}$ are positive for positive $\epsilon^{\pm}_k$, i.e., $\Gamma^{\mathtt{L}}_{k\pm} > 0$ for $\epsilon^{\pm}_k > 0$. Then, as time t becomes infinite $(t \to \infty)$, $n^{\alpha}_k(t)$ and $n^{\beta}_k(t)$ approach the finite values

$$n^{\alpha}_k(\infty) = \frac{\bar{n}^+_k\,\Gamma^{\mathtt{L}}_{k-}(\Gamma^{\mathtt{L}}_{k+} + \Psi'_k \sinh^2 2\theta_k) + (\bar{n}^-_k + 1)(\Gamma^{\mathtt{L}}_{k-} + \Psi'_k \sinh^2 2\theta_k)\Psi'_k \sinh^2 2\theta_k}{\Gamma^{\mathtt{L}}_{k+}\Gamma^{\mathtt{L}}_{k-} - (\Psi'_k)^2 \sinh^4 2\theta_k}, \tag{A.49a}$$

$$n^{\beta}_k(\infty) = \frac{\bar{n}^-_k\,\Gamma^{\mathtt{L}}_{k+}(\Gamma^{\mathtt{L}}_{k-} + \Psi'_k \sinh^2 2\theta_k) + (\bar{n}^+_k + 1)(\Gamma^{\mathtt{L}}_{k+} + \Psi'_k \sinh^2 2\theta_k)\Psi'_k \sinh^2 2\theta_k}{\Gamma^{\mathtt{L}}_{k+}\Gamma^{\mathtt{L}}_{k-} - (\Psi'_k)^2 \sinh^4 2\theta_k}, \tag{A.49b}$$

which are derived from Eqs. (A.39c) and (A.40c) in the infinite limit $(t \to \infty)$, where we have put $\bar{n}^{\pm}_k = \bar{n}(\epsilon^{\pm}_k)$.

The equations of motion for the quasi-particle operators $\lambda_k(t)$ and $\xi_k(t)$ can be derived, by performing the transformations (A.33), (A.37) and their tilde conjugates, by using the thermal-state conditions (A.29) and their tilde conjugates, and by considering the assumption that the phonon correlation function (2.24c) is real, as follows,

$$\begin{aligned}
&(d/dt)\, Z^{\alpha}_k(t)^{1/2}\langle 1_{\mathtt{S}}|\lambda_k(t) = (d/dt)\,\langle 1_{\mathtt{S}}|\,\alpha_k(t) = \langle 1_{\mathtt{S}}|\,U^{-1}(t)\,[\,i\,\hat{\mathcal{H}}_{\mathtt{S}0} - C^{(2)}, \alpha_k\,]\,U(t),\\
&= \langle 1_{\mathtt{S}}|\big\{-i\,\epsilon^+_k\,\alpha_k(t) - \alpha_k(t)\{S(\phi^{-+}_k(\epsilon^+_k) - \phi^{+-}_k(\epsilon^+_k)^*) - S\,(\phi^{+-}_k(-\epsilon^+_k) - \phi^{-+}_k(-\epsilon^+_k)^*)\}/2\\
&\qquad - \alpha_k(t)\{S\,(\phi^{-+}_k(\epsilon^+_k) - \phi^{+-}_k(\epsilon^+_k)^*) + S\,(\phi^{+-}_k(-\epsilon^+_k) - \phi^{-+}_k(-\epsilon^+_k)^*)\}\cosh 2\theta_k/2\\
&\qquad - \beta^{\dagger\dagger}_k(t)\{S\,(\phi^{+-}_k(-\epsilon^-_k) - \phi^{-+}_k(-\epsilon^-_k)^*)^* + S\,(\phi^{-+}_k(\epsilon^-_k) - \phi^{+-}_k(\epsilon^-_k)^*)^*\}\sinh 2\theta_k/2\\
&\qquad - \phi^{zz}_k(0)\,\alpha_k(t)\cosh^2 2\theta_k/2 - \phi^{zz}_k(0)\,\alpha_k(t)/2 + \phi^{zz}_k(\epsilon^+_k + \epsilon^-_k)\,\alpha_k(t)\sinh^2 2\theta_k/2\\
&\qquad - \phi^{zz}_k(0)\,\beta^{\dagger\dagger}_k(t)\sinh 2\theta_k\cosh 2\theta_k/2 + \phi^{zz}_k(\epsilon^+_k + \epsilon^-_k)^*\,\beta^{\dagger\dagger}_k(t)\sinh 2\theta_k\cosh 2\theta_k/2\big\},
\end{aligned} \tag{A.50a}$$

$$\begin{aligned}
&= \big\{-i\,\epsilon^+_k - \{S\,(\Phi^+_k(\epsilon^+_k) + S\Phi^-_k(\epsilon^+_k))\cosh 2\theta_k + S\,(\Phi^+_k(\epsilon^+_k) - \Phi^-_k(\epsilon^+_k))\}/2 - \Psi^0_k\cosh^2 2\theta_k/2\\
&\quad - \Psi^0_k/2 + \Psi_k\sinh^2 2\theta_k/2\big\}\,Z^{\alpha}_k(t)^{1/2}\,\langle 1_{\mathtt{S}}|\,\lambda_k(t) - \big\{S(\Phi^-_k(\epsilon^-_k)^* + S\Phi^+_k(\epsilon^-_k)^*)\sinh 2\theta_k/2\\
&\quad + (\Psi^0_k - \Psi^*_k)\sinh 2\theta_k\cosh 2\theta_k/2\big\}\,Z^{\beta}_k(t)^{1/2}\,\langle 1_{\mathtt{S}}|\,\tilde{\xi}_k(t),
\end{aligned} \tag{A.50b}$$

$$\begin{aligned}
&(d/dt)\, Z^{\beta}_k(t)^{1/2}\langle 1_{\mathtt{S}}|\,\xi_k(t) = (d/dt)\,\langle 1_{\mathtt{S}}|\,\beta_k(t) = \langle 1_{\mathtt{S}}|\,U^{-1}(t)\,[\,i\,\hat{\mathcal{H}}_{\mathtt{S}0} - C^{(2)}, \beta_k\,]\,U(t),\\
&= \langle 1_{\mathtt{S}}|\big\{-i\,\epsilon^-_k\,\beta_k(t) + \beta_k(t)\{S(\phi^{+-}_k(-\epsilon^-_k) - \phi^{-+}_k(-\epsilon^-_k)^*) - S\,(\phi^{-+}_k(\epsilon^-_k) - \phi^{+-}_k(\epsilon^-_k)^*)\}/2\\
&\qquad - \beta_k(t)\{S\,(\phi^{+-}_k(-\epsilon^-_k) - \phi^{-+}_k(-\epsilon^-_k)^*) + S\,(\phi^{-+}_k(\epsilon^-_k) - \phi^{+-}_k(\epsilon^-_k)^*)\}\cosh 2\theta_k/2\\
&\qquad - \alpha^{\dagger\dagger}_k(t)\{S\,(\phi^{-+}_k(\epsilon^+_k) - \phi^{+-}_k(\epsilon^+_k)^*)^* + S\,(\phi^{+-}_k(-\epsilon^+_k) - \phi^{-+}_k(-\epsilon^+_k)^*)^*\}\sinh 2\theta_k/2\\
&\qquad - \phi^{zz}_k(0)\,\beta_k(t)\cosh^2 2\theta_k/2 - \phi^{zz}_k(0)\,\beta_k(t)/2 + \phi^{zz}_k(\epsilon^+_k + \epsilon^-_k)\,\beta_k(t)\sinh^2 2\theta_k/2\\
&\qquad - \phi^{zz}_k(0)\,\alpha^{\dagger\dagger}_k(t)\sinh 2\theta_k\cosh 2\theta_k/2 + \phi^{zz}_k(\epsilon^+_k + \epsilon^-_k)^*\,\alpha^{\dagger\dagger}_k(t)\sinh 2\theta_k\cosh 2\theta_k/2\big\},
\end{aligned} \tag{A.51a}$$

$$\begin{aligned}
&= \big\{-i\,\epsilon^-_k - \{S\,(\Phi^-_k(\epsilon^-_k) + \Phi^+_k(\epsilon^-_k))\cosh 2\theta_k - S(\Phi^-_k(\epsilon^-_k) - \Phi^+_k(\epsilon^-_k))\}/2 - \Psi^0_k\cosh^2 2\theta_k/2\\
&\quad - \Psi^0_k/2 + \Psi_k\sinh^2 2\theta_k/2\big\}\,Z^{\beta}_k(t)^{1/2}\,\langle 1_{\mathtt{S}}|\,\xi_k(t) - \big\{S\,(\Phi^+_k(\epsilon^+_k)^* + \Phi^-_k(\epsilon^+_k)^*)\sinh 2\theta_k/2\\
&\quad + (\Psi^0_k - \Psi^*_k)\sinh 2\theta_k\cosh 2\theta_k/2\big\}\,Z^{\alpha}_k(t)^{1/2}\,\langle 1_{\mathtt{S}}|\,\tilde{\lambda}_k(t),
\end{aligned} \tag{A.51b}$$

where $\Phi^+_k(\epsilon)$, $\Phi^-_k(\epsilon)$ and Ψ_k are given by (A.42) – (A.44). The above equations can be rewritten as

$$(d/dt)Z^{\alpha}_k(t)^{1/2}\langle 1_{\mathtt{S}}|\lambda_k(t) = \{-i\,\epsilon^+_k - \Gamma_{k+}\}Z^{\alpha}_k(t)^{1/2}\langle 1_{\mathtt{S}}|\lambda_k(t) - \Delta^*_{k-}Z^{\beta}_k(t)^{1/2}\langle 1_{\mathtt{S}}|\,\tilde{\xi}_k(t), \tag{A.52a}$$

$$(d/dt)Z^{\beta}_k(t)^{1/2}\langle 1_{\mathtt{S}}|\,\tilde{\xi}_k(t) = \{i\,\epsilon^-_k - \Gamma^*_{k-}\}Z^{\beta}_k(t)^{1/2}\langle 1_{\mathtt{S}}|\,\tilde{\xi}_k(t) - \Delta_{k+}Z^{\alpha}_k(t)^{1/2}\langle 1_{\mathtt{S}}|\lambda_k(t). \tag{A.52b}$$

where we have put for brevity as

$$\begin{aligned}
\Gamma_{k\pm} &= \{S\,(\Phi^{\pm}_k(\epsilon^{\pm}_k) + \Phi^{\mp}_k(\epsilon^{\pm}_k))\cosh 2\theta_k \pm S\,(\Phi^{\pm}_k(\epsilon^{\pm}_k) - \Phi^{\mp}_k(\epsilon^{\pm}_k))\}/2\\
&\quad - \Psi_k\sinh^2 2\theta_k/2 + \Psi^0_k\cdot(\cosh^2 2\theta_k + 1)/2,
\end{aligned} \tag{A.53a}$$

$$\Delta_{k\pm} = S\,(\Phi^{\pm}_k(\epsilon^{\pm}_k) + \Phi^{\mp}_k(\epsilon^{\pm}_k))\sinh 2\theta_k/2 + (\Psi^0_k - \Psi_k)\sinh 2\theta_k\cosh 2\theta_k/2, \tag{A.53b}$$

with $\Phi^+_k(\epsilon)$, $\Phi^-_k(\epsilon)$ and Ψ_k defined by (A.42), (A.43) and (A.44), respectively. Here, we have put

$$\Psi^0_k = \phi^{zz}_k(0) = \int_0^{\infty} d\tau \sum_{\nu} g^2_{2\nu}\langle 1_{\mathtt{R}}|\Delta(R^{\dagger}_{k\nu}(\tau)R_{k\nu}(\tau))\Delta(R^{\dagger}_{k\nu}R_{k\nu})|\rho_{\mathtt{R}}\rangle, \tag{A.54}$$

which is real according to the assumption that the phonon correlation function (2.24c) is real. The solutions of Eqs. (A.52a) and (A.52b) can be written as

$$Z_k^\alpha(t)^{1/2}\,\langle 1_{\rm S}|\lambda_k(t) = Z_k^\alpha(\tau)^{1/2}\,\langle 1_{\rm S}|\lambda_k(\tau)\exp\{(-\,i\,\epsilon_k^+ - \Gamma_{k+})(t-\tau)\}$$
$$\qquad - \int_\tau^t dt_1 \exp\{(-\,i\,\epsilon_k^+ - \Gamma_{k+})(t-t_1)\}\,\Delta_{k-}^*\, Z_k^\beta(t_1)^{1/2}\,\langle 1_{\rm S}|\,\tilde{\xi}_k(t_1), \tag{A.55a}$$
$$Z_k^\beta(t)^{1/2}\,\langle 1_{\rm S}|\,\tilde{\xi}_k(t) = Z_k^\beta(\tau)^{1/2}\,\langle 1_{\rm S}|\,\tilde{\xi}_k(\tau)\exp\{(i\,\epsilon_k^- - \Gamma_{k-}^*)(t-\tau)\}$$
$$\qquad - \int_\tau^t dt_1 \exp\{(i\,\epsilon_k^- - \Gamma_{k-}^*)(t-t_1)\}\,\Delta_{k+}\, Z_k^\alpha(t_1)^{1/2}\,\langle 1_{\rm S}|\lambda_k(t_1), \tag{A.55b}$$

from which we can obtain the approximate solutions as in Ref. [21]. Thus, we can obtain the forms of the quasi-particle operators, which are valid up to second order in powers of the spin-phonon interaction, as

$$\langle 1_{\rm S}|\lambda_k(t) = Z_k^\alpha(t)^{-1/2}\, Z_k^\alpha(\tau)^{1/2}\exp\{(-\,i\,\epsilon_k^+ - \Gamma_{k+})(t-\tau)\}\langle 1_{\rm S}|\lambda_k(\tau)$$
$$\qquad + \Delta_{k-}^*\frac{Z_k^\beta(\tau)^{1/2}}{Z_k^\alpha(t)^{1/2}}\cdot\frac{\exp\{(-\,i\,\epsilon_k^+ - \Gamma_{k+})(t-\tau)\} - \exp\{(i\,\epsilon_k^- - \Gamma_{k-}^*)(t-\tau)\}}{i\,(\epsilon_k^+ + \epsilon_k^-) + \Gamma_{k+} - \Gamma_{k-}^*}\langle 1_{\rm S}|\,\tilde{\xi}_k(\tau), \tag{A.56a}$$
$$\langle 1_{\rm S}|\,\tilde{\xi}_k(t) = Z_k^\beta(t)^{-1/2}\, Z_k^\beta(\tau)^{1/2}\exp\{(i\,\epsilon_k^- - \Gamma_{k-}^*)(t-\tau)\}\langle 1_{\rm S}|\,\tilde{\xi}_k(\tau)$$
$$\qquad + \Delta_{k+}\frac{Z_k^\alpha(\tau)^{1/2}}{Z_k^\beta(t)^{1/2}}\cdot\frac{\exp\{(-\,i\,\epsilon_k^+ - \Gamma_{k+})(t-\tau)\} - \exp\{(i\,\epsilon_k^- - \Gamma_{k-}^*)(t-\tau)\}}{i\,(\epsilon_k^+ + \epsilon_k^-) + \Gamma_{k+} - \Gamma_{k-}^*}\langle 1_{\rm S}|\lambda_k(\tau). \tag{A.56b}$$

Rewriting the quasi-particle forms (A.56a) and (A.56b) for $\tau = 0$ by putting $\lambda_k = \lambda_k(0)$ and $\xi_k = \xi_k(0)$, we have

$$\langle 1_{\rm S}|\lambda_k(t) = Z_k^\alpha(t)^{-1/2}\, Z_k^\alpha(0)^{1/2}\exp\{(-\,i\,\epsilon_k^+ - \Gamma_{k+})\,t\,\}\langle 1_{\rm S}|\lambda_k$$
$$\qquad + \Delta_{k-}^*\frac{Z_k^\beta(0)^{1/2}}{Z_k^\alpha(t)^{1/2}}\cdot\frac{\exp\{(-\,i\,\epsilon_k^+ - \Gamma_{k+})\,t\,\} - \exp\{(i\,\epsilon_k^- - \Gamma_{k-}^*)\,t\,\}}{i\,(\epsilon_k^+ + \epsilon_k^-) + \Gamma_{k+} - \Gamma_{k-}^*}\langle 1_{\rm S}|\,\tilde{\xi}_k\,, \tag{A.57a}$$
$$\langle 1_{\rm S}|\,\tilde{\xi}_k(t) = Z_k^\beta(t)^{-1/2}\, Z_k^\beta(0)^{1/2}\exp\{(i\,\epsilon_k^- - \Gamma_{k-}^*)\,t\,\}\langle 1_{\rm S}|\,\tilde{\xi}_k$$
$$\qquad + \Delta_{k+}\frac{Z_k^\alpha(0)^{1/2}}{Z_k^\beta(t)^{1/2}}\cdot\frac{\exp\{(-\,i\,\epsilon_k^+ - \Gamma_{k+})\,t\,\} - \exp\{(i\,\epsilon_k^- - \Gamma_{k-}^*)\,t\,\}}{i\,(\epsilon_k^+ + \epsilon_k^-) + \Gamma_{k+} - \Gamma_{k-}^*}\langle 1_{\rm S}|\lambda_k\,. \tag{A.57b}$$

These formulas are useful for the perturbation calculations of correlation functions, susceptibilities, et al.

From the above quasi-particle forms (A.57a) and (A.57b), we can obtain the quasi-particle correlation forms as

$$\langle 1_{\rm S}|\lambda_k(t)\,\lambda_k^\ddagger|\rho_0\rangle = Z_k^\alpha(t)^{-1/2}\, Z_k^\alpha(0)^{1/2}\exp\{-\,i\,(\epsilon_k^+ + \Gamma_{k+}'')\,t - \Gamma_{k+}'\,t\,\}, \tag{A.58a}$$
$$\langle 1_{\rm S}|\,\xi_k(t)\,\xi_k^\ddagger|\rho_0\rangle = Z_k^\beta(t)^{-1/2}\, Z_k^\beta(0)^{1/2}\exp\{-\,i\,(\epsilon_k^- + \Gamma_{k-}'')\,t - \Gamma_{k-}'\,t\,\}, \tag{A.58b}$$
$$\langle 1_{\rm S}|\,\tilde{\xi}_k(t)\,\lambda_k^\ddagger|\rho_0\rangle = \frac{\exp\{-\,i\,(\epsilon_k^+ + \Gamma_{k+}'')\,t - \Gamma_{k+}'\,t\,\} - \exp\{i\,(\epsilon_k^- + \Gamma_{k-}'')\,t - \Gamma_{k-}'\,t\,\}}{i\,(\epsilon_k^+ + \epsilon_k^- + \Gamma_{k+}'' + \Gamma_{k-}'') + \Gamma_{k+}' - \Gamma_{k-}'}$$
$$\qquad \times\, Z_k^\beta(t)^{-1/2}\, Z_k^\alpha(0)^{1/2}\,(\Delta_{k+}' + i\,\Delta_{k+}''), \tag{A.58c}$$
$$\langle 1_{\rm S}|\tilde{\lambda}_k(t)\,\xi_k^\ddagger|\rho_0\rangle = \frac{\exp\{-\,i\,(\epsilon_k^- + \Gamma_{k-}'')\,t - \Gamma_{k-}'\,t\,\} - \exp\{i\,(\epsilon_k^+ + \Gamma_{k+}'')\,t - \Gamma_{k+}'\,t\,\}}{i\,(\epsilon_k^+ + \epsilon_k^- + \Gamma_{k+}'' + \Gamma_{k-}'') + \Gamma_{k-}' - \Gamma_{k+}'}$$
$$\qquad \times\, Z_k^\alpha(t)^{-1/2}\, Z_k^\beta(0)^{1/2}\,(\Delta_{k-}' + i\,\Delta_{k-}''), \tag{A.58d}$$

with $\lambda_k^\ddagger = \lambda_k^\ddagger(0)$ and $\xi_k^\ddagger = \xi_k^\ddagger(0)$, where $\Gamma_{k\pm}'$, $\Delta_{k\pm}'$ and $\Gamma_{k\pm}''$, $\Delta_{k\pm}''$ are the real parts and the imaginary parts of $\Gamma_{k\pm}$ and $\Delta_{k\pm}$, which are defined by (A.53a) and (A.53b), respectively, and are given by

$$\Gamma_{k\pm}' = S\,\Phi_k^\pm(\epsilon_k^\pm)'(\cosh 2\theta_k \pm 1)/2 + S\,\Phi_k^\mp(\epsilon_k^\pm)'(\cosh 2\theta_k \mp 1)/2$$
$$\qquad - (\Psi_k'/2)\sinh^2 2\theta_k + (\Psi_k^0/2)(\cosh^2 2\theta_k + 1), \tag{A.59a}$$
$$\Gamma_{k\pm}'' = S\,\Phi_k^\pm(\epsilon_k^\pm)''(\cosh 2\theta_k \pm 1)/2 + S\,\Phi_k^\mp(\epsilon_k^\pm)''(\cosh 2\theta_k \mp 1)/2 - (\Psi_k''/2)\sinh^2 2\theta_k, \tag{A.59b}$$
$$\Delta_{k\pm}' = S\,(\Phi_k^\pm(\epsilon_k^\pm)' + \Phi_k^\mp(\epsilon_k^\pm)')\sinh 2\theta_k/2 + (\Psi_k^0 - \Psi_k')\sinh 2\theta_k\cosh 2\theta_k/2, \tag{A.59c}$$
$$\Delta_{k\pm}'' = S\,(\Phi_k^\pm(\epsilon_k^\pm)'' + \Phi_k^\mp(\epsilon_k^\pm)'')\sinh 2\theta_k/2 - \Psi_k''\sinh 2\theta_k\cosh 2\theta_k/2. \tag{A.59d}$$

Considering that $\Phi_k^\pm(\epsilon_k^\pm)'$ is positive for positive $\epsilon_k^\pm$, i.e., $\Phi_k^\pm(\epsilon_k^\pm)' > 0$ for $\epsilon_k^\pm > 0$, as shown in Appendix A of Ref. [21], and that Ψ_k^0 is non-negative, i.e., $\Psi_k^0 \geq 0$, as shown in Ref. [29, 30], we notice from (A.44) and (A.54) that $\Gamma_{k\pm}'$ are positive for positive $\epsilon_k^\pm$, i.e.,

$$\Gamma_{k\pm}' \geq S\,\Phi_k^\pm(\epsilon_k^\pm)'(\cosh 2\theta_k \pm 1)/2 + S\,\Phi_k^\mp(\epsilon_k^\pm)'(\cosh 2\theta_k \mp 1)/2 + \Psi_k^0 > 0, \qquad \text{for} \qquad \epsilon_k^\pm > 0. \tag{A.60}$$

The quasi-particle correlation forms (A.58a) and (A.58b) for the semi-free field show that the λ quasi-particle with the wave-number k has the energy $\hbar(\epsilon_k^+ + \Gamma''_{k+})$ and decays exponentially with the life-time $(\Gamma'_{k+})^{-1}$, that the ξ quasi-particle with the wave-number k has the energy $\hbar(\epsilon_k^- + \Gamma''_{k-})$ and decays exponentially with the life-time $(\Gamma'_{k-})^{-1}$. The quasi-particle correlation forms (A.58c) and (A.58d) for the semi-free field show that the λ quasi-particle and the ξ quasi-particle change to the $\tilde{\xi}$ quasi-particle and the $\tilde{\lambda}$ quasi-particle, respectively, through the spin-phonon interaction.

B Form of the interference thermal state $|D^{(2)}_{S_k^-}[\omega]\rangle$

In this Appendix, a form of the interference thermal state $|D^{(2)}_{S_k^-}[\omega]\rangle$ given by (3.8) is derived. The interference thermal state $|D^{(2)}_{S_k^-}[\omega]\rangle$ given by (3.8) can be expressed by substituting the spin-phonon interaction (2.21) into (3.8) and by using the free spin-wave Hamiltonian (2.19), the axioms (A.2), (A.8) and their tilde conjugates, and the assumptions (2.23a), (2.23b) and (2.24a) – (2.24c), as

$$
\begin{aligned}
|D^{(2)}_{S_k^-}[\omega]\rangle = {} & \frac{i\,\gamma\,S\sqrt{S}}{4\sqrt{2}} \int_0^\infty d\tau \int_0^\tau ds \sum_\nu |g_{1\nu}|^2 \,(\cosh\theta_k - \sinh\theta_k) \\
& \times \big\{ (\langle 1_{\mathrm{R}}|R_{k\nu}(\tau)R^\dagger_{k\nu}|\rho_{\mathrm{R}}\rangle - \langle 1_{\mathrm{R}}|R^\dagger_{k\nu}R_{k\nu}(\tau)|\rho_{\mathrm{R}}\rangle)\exp(i\,\omega\,s) \\
& \quad \times \big\{ \{(\cosh 2\theta_k + 1)\,(\alpha_k^\dagger - \tilde{\alpha}_k) - \sinh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)\}|\rho_0\rangle \exp(i\,\epsilon_k^+\,\tau - i\,\epsilon_k^+\,s) \\
& \quad\quad + \{\sinh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k) - (\cosh 2\theta_k - 1)\,(\beta_k - \tilde{\beta}_k^\dagger)\}|\rho_0\rangle \exp(-\,i\,\epsilon_k^-\,\tau + i\,\epsilon_k^-\,s) \big\} \\
& \quad + (\langle 1_{\mathrm{R}}|R^\dagger_{k\nu}(\tau)R_{k\nu}|\rho_{\mathrm{R}}\rangle - \langle 1_{\mathrm{R}}|R_{k\nu}R^\dagger_{k\nu}(\tau)|\rho_{\mathrm{R}}\rangle)\exp(i\,\omega\,s) \\
& \quad \times \big\{ \{(\cosh 2\theta_k - 1)\,(\alpha_k^\dagger - \tilde{\alpha}_k) - \sinh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)\}|\rho_0\rangle \exp(i\,\epsilon_k^+\,\tau - i\,\epsilon_k^+\,s) \\
& \quad\quad + \{\sinh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k) - (\cosh 2\theta_k + 1)\,(\beta_k - \tilde{\beta}_k^\dagger)\}|\rho_0\rangle \exp(-\,i\,\epsilon_k^-\,\tau + i\,\epsilon_k^-\,s) \big\} \big\} \\
& + \frac{i\,\gamma\sqrt{S}}{2\sqrt{2}} \int_0^\infty d\tau \int_0^\tau ds \sum_\nu g^2_{2\nu}(\cosh\theta_k - \sinh\theta_k)\langle 1_{\mathrm{R}}|\Delta(R^\dagger_{k\nu}(\tau)R_{k\nu}(\tau))\Delta(R^\dagger_{k\nu}R_{k\nu})|\rho_{\mathrm{R}}\rangle \exp(i\,\omega\,s) \\
& \quad \times \big\{ \sinh 2\theta_k \cosh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle \exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\,\tau - i\,\epsilon_k^+\,s\} \\
& \quad\quad + \sinh 2\theta_k \cosh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle \exp\{-\,i\,(\epsilon_k^+ + \epsilon_k^-)\,\tau + i\,\epsilon_k^-\,s\} \\
& \quad\quad + (\cosh^2 2\theta_k + 1)\,(\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle \exp(-\,i\,\epsilon_k^+\,s) + (\cosh^2 2\theta_k + 1)\,(\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle \exp(i\,\epsilon_k^-\,s) \\
& \quad\quad + (\alpha_k\beta_k + \alpha_k^\dagger\beta_k^\dagger - \tilde{\alpha}_k\tilde{\beta}_k - \tilde{\alpha}_k^\dagger\tilde{\beta}_k^\dagger)\sinh 2\theta_k \\
& \quad\quad\quad \times \big\{ \sinh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle \exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\,\tau - i\,\epsilon_k^+\,s\} \\
& \quad\quad\quad\quad - \sinh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle \exp\{-\,i\,(\epsilon_k^+ + \epsilon_k^-)\,\tau + i\,\epsilon_k^-\,s\} \\
& \quad\quad\quad\quad - \cosh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle \exp(-\,i\,\epsilon_k^+\,s) + \cosh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle \exp(i\,\epsilon_k^-\,s) \big\} \big\},
\end{aligned} \tag{B.1}
$$

with $\Delta(R^\dagger_{k\nu}(t)R_{k\nu}(t)) = R^\dagger_{k\nu}(t)R_{k\nu}(t) - \langle 1_{\mathrm{R}}|R^\dagger_{k\nu}R_{k\nu}|\rho_{\mathrm{R}}\rangle$ and $\Delta(R^\dagger_{k\nu}R_{k\nu}) = R^\dagger_{k\nu}R_{k\nu} - \langle 1_{\mathrm{R}}|R^\dagger_{k\nu}R_{k\nu}|\rho_{\mathrm{R}}\rangle$, where we have ignored the higher-order parts in the spin-wave approximation, and have used the assumption that the phonon correlation function given by (2.24c) is real. Here, we have used the relations $\alpha_k^\dagger\alpha_k|\rho_0\rangle = \tilde{\alpha}_k^\dagger\tilde{\alpha}_k|\rho_0\rangle$ and $\beta_k^\dagger\beta_k|\rho_0\rangle = \tilde{\beta}_k^\dagger\tilde{\beta}_k|\rho_0\rangle$, which are led from the thermal-state conditions (A.30) and their tilde conjugates. The above form of interference thermal state $|D^{(2)}_{S_k^-}[\omega]\rangle$ can be written by using the correlation functions $\phi_k^{+-}(\epsilon)$, $\phi_k^{-+}(\epsilon)$ and $\phi_k^{zz}(\epsilon)$ defined by (A.25a) – (A.25c), as

$$
\begin{aligned}
|D^{(2)}_{S_k^-}[\omega]\rangle = {} & \frac{\gamma\,S\sqrt{S}}{2\sqrt{2}} \big\{ (\cosh\theta_k - \sinh\theta_k)\{(\cosh 2\theta_k + 1)\,(\alpha_k^\dagger - \tilde{\alpha}_k) - \sinh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)\}|\rho_0\rangle \\
& \quad \times \{(\phi_k^{-+}(\omega) - \phi_k^{+-}(\omega)^*) - (\phi_k^{-+}(\epsilon_k^+) - \phi_k^{+-}(\epsilon_k^+)^*)\}/(\omega - \epsilon_k^+) \\
& + (\cosh\theta_k - \sinh\theta_k)\{\sinh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k) - (\cosh 2\theta_k - 1)\,(\beta_k - \tilde{\beta}_k^\dagger)\}|\rho_0\rangle \\
& \quad \times \{(\phi_k^{-+}(\omega) - \phi_k^{+-}(\omega)^*) - (\phi_k^{-+}(-\,\epsilon_k^-) - \phi_k^{+-}(-\,\epsilon_k^-)^*)\}/(\omega + \epsilon_k^-) \\
& + (\cosh\theta_k - \sinh\theta_k)\{(\cosh 2\theta_k - 1)\,(\alpha_k^\dagger - \tilde{\alpha}_k) - \sinh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)\}|\rho_0\rangle \\
& \quad \times \{(\phi_k^{+-}(-\,\omega) - \phi_k^{-+}(-\,\omega)^*) - (\phi_k^{+-}(-\,\epsilon_k^+) - \phi_k^{-+}(-\,\epsilon_k^+)^*)\}/(\omega - \epsilon_k^+) \\
& + (\cosh\theta_k - \sinh\theta_k)\{\sinh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k) - (\cosh 2\theta_k + 1)\,(\beta_k - \tilde{\beta}_k^\dagger)\}|\rho_0\rangle \\
& \quad \times \{(\phi_k^{+-}(-\,\omega) - \phi_k^{-+}(-\,\omega)^*) - (\phi_k^{+-}(\epsilon_k^-) - \phi_k^{-+}(\epsilon_k^-)^*)\}/(\omega + \epsilon_k^-)\big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\gamma\sqrt{S}}{2\sqrt{2}}\,(\cosh\theta_k-\sinh\theta_k)\\
&\quad\times\big\{\{(\cosh^2 2\theta_k+1)\,(\alpha_k^\dagger-\tilde{\alpha}_k)|\rho_0\rangle\{\phi_k^{zz}(\omega-\epsilon_k^+)-\phi_k^{zz}(0)\}\\
&\qquad+\sinh 2\theta_k\cosh 2\theta_k\,(\beta_k-\tilde{\beta}_k^\dagger)|\rho_0\rangle\{\phi_k^{zz}(\omega+\epsilon_k^-)-\phi_k^{zz}(\epsilon_k^++\epsilon_k^-)\}\}/(\omega-\epsilon_k^+)\\
&\quad+\{\sinh 2\theta_k\cosh 2\theta_k\,(\alpha_k^\dagger-\tilde{\alpha}_k)|\rho_0\rangle\{\phi_k^{zz}(\omega-\epsilon_k^+)-\phi_k^{zz}(-\epsilon_k^+-\epsilon_k^-)\}\\
&\qquad+(\cosh^2 2\theta_k+1)\,(\beta_k-\tilde{\beta}_k^\dagger)|\rho_0\rangle\{\phi_k^{zz}(\omega+\epsilon_k^-)-\phi_k^{zz}(0)\}\}/(\omega+\epsilon_k^-)\\
&\quad+(\alpha_k\beta_k+\alpha_k^\dagger\beta_k^\dagger-\tilde{\alpha}_k\tilde{\beta}_k-\tilde{\alpha}_k^\dagger\tilde{\beta}_k^\dagger)\sinh 2\theta_k\\
&\qquad\times\big\{\{\sinh 2\theta_k\,(\beta_k-\tilde{\beta}_k^\dagger)|\rho_0\rangle\{\phi_k^{zz}(\omega+\epsilon_k^-)-\phi_k^{zz}(\epsilon_k^++\epsilon_k^-)\}\\
&\qquad\quad-\cosh 2\theta_k\,(\alpha_k^\dagger-\tilde{\alpha}_k)|\rho_0\rangle\{\phi_k^{zz}(\omega-\epsilon_k^+)-\phi_k^{zz}(0)\}\}/(\omega-\epsilon_k^+)\\
&\qquad+\{\cosh 2\theta_k\,(\beta_k-\tilde{\beta}_k^\dagger)|\rho_0\rangle\{\phi_k^{zz}(\omega+\epsilon_k^-)-\phi_k^{zz}(0)\}\\
&\qquad\quad-\sinh 2\theta_k\,(\alpha_k^\dagger-\tilde{\alpha}_k)|\rho_0\rangle\{\phi_k^{zz}(\omega-\epsilon_k^+)-\phi_k^{zz}(-\epsilon_k^+-\epsilon_k^-)\}\}/(\omega+\epsilon_k^-)\big\}\big\}. \qquad \text{(B.2)}
\end{aligned}
$$

C Calculation of corresponding interference terms $X_{k1(2)}^{\alpha(\beta)}(\omega)$

In this Appendix, the forms of the corresponding interference terms $X_{k1(2)}^{\alpha(\beta)}(\omega)$ defined by (3.15a) – (3.16b), are derived. In order to deal with the fractions in the calculations of $X_{k1(2)}^{\alpha(\beta)}(\omega)$ defined by (3.15a) – (3.16b), we use the following forms for $\Phi_k^{\pm}(\epsilon)$ defined by (A.42) and (A.43) with the phonon correlation functions given by (4.1a) and (4.1b):

$$
\Phi_k^+(\epsilon)=\frac{1}{2}\int_0^\infty d\tau\sum_\nu|g_{1\nu}|^2\,\langle 1_{\mathrm{R}}|[R_{k\nu}(\tau),R_{k\nu}^\dagger]|\rho_{\mathrm{R}}\rangle\exp(i\,\epsilon\,\tau)=\frac{g_1^2/2}{-\,i\,(\epsilon-\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}}, \qquad \text{(C.1)}
$$

$$
\Phi_k^-(\epsilon)=\frac{1}{2}\int_0^\infty d\tau\sum_\nu|g_{1\nu}|^2\,\langle 1_{\mathrm{R}}|[R_{k\nu}^\dagger(\tau),R_{k\nu}]|\rho_{\mathrm{R}}\rangle\exp(i\,\epsilon\,\tau)=\frac{-\,g_1^2/2}{-\,i\,(\epsilon+\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}}. \qquad \text{(C.2)}
$$

The forms of the corresponding interference terms $X_{k1(2)}^{\alpha(\beta)}(\omega)$ defined by (3.15a) – (3.16b), are derived using (C.1), (C.2) and (4.5) – (4.7) as follows,

$$
\begin{aligned}
X_{k1}^{\alpha}(\omega)&=\langle 1_{\mathrm{S}}|\,\alpha_k\,|D_{k1}^{(2)}[\omega]\rangle/(2\,(\omega-\epsilon_k^+))=X_{k1}^{\alpha}(\omega)'+i\,X_{k1}^{\alpha}(\omega)'',\\
&=\frac{i\,(g_1^2/4)\,S\,(\cosh 2\theta_k+1)}{\{-\,i\,(\omega-\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}\{-\,i\,(\epsilon_k^+-\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}}-\frac{i\,(g_1^2/4)\,S\,(\cosh 2\theta_k-1)\}}{\{-\,i\,(\omega+\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}\{-\,i\,(\epsilon_k^++\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}}\\
&\quad-g_2^2\sinh^2 2\theta_k\,\frac{i\,\bar{n}(\omega_{\mathrm{R}k})\{\bar{n}(\omega_{\mathrm{R}k})+1\}}{2\,\{-\,i\,(\omega+\epsilon_k^-)+2\gamma_{\mathrm{R}k}\}\{-\,i\,(\epsilon_k^++\epsilon_k^-)+2\gamma_{\mathrm{R}k}\}}\\
&\quad+g_2^2\,(\cosh^2 2\theta_k+1)\,\frac{i\,\bar{n}(\omega_{\mathrm{R}k})\{\bar{n}(\omega_{\mathrm{R}k})+1\}}{4\,\gamma_{\mathrm{R}k}\{-\,i\,(\omega-\epsilon_k^+)+2\gamma_{\mathrm{R}k}\}}, \qquad \text{(C.3a)}\\
&=g_1^2\,S\,\frac{-\,\gamma_{\mathrm{R}k}\,(\omega+\epsilon_k^+-2\,\omega_{\mathrm{R}k})+i\,\{(\gamma_{\mathrm{R}k})^2-(\omega-\omega_{\mathrm{R}k})(\epsilon_k^+-\omega_{\mathrm{R}k})\}}{4\,\{(\omega-\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}\{(\epsilon_k^+-\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}}(\cosh 2\theta_k+1)\\
&\quad+g_1^2\,S\,\frac{\gamma_{\mathrm{R}k}\,(\omega+\epsilon_k^++2\,\omega_{\mathrm{R}k})-i\,\{(\gamma_{\mathrm{R}k})^2-(\omega+\omega_{\mathrm{R}k})(\epsilon_k^++\omega_{\mathrm{R}k})\}}{4\,\{(\omega+\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}\{(\epsilon_k^++\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}}(\cosh 2\theta_k-1)\\
&\quad+g_2^2\,\frac{2\,\gamma_{\mathrm{R}k}\,(\omega+\epsilon_k^++2\,\epsilon_k^-)-i\,\{4\,(\gamma_{\mathrm{R}k})^2-(\omega+\epsilon_k^-)(\epsilon_k^++\epsilon_k^-)\}}{2\,\{(\omega+\epsilon_k^-)^2+4\,(\gamma_{\mathrm{R}k})^2\}\{(\epsilon_k^++\epsilon_k^-)^2+4\,(\gamma_{\mathrm{R}k})^2\}}\,\bar{n}(\omega_{\mathrm{R}k})\{\bar{n}(\omega_{\mathrm{R}k})+1\}\sinh^2 2\theta_k\\
&\quad+g_2^2\,\frac{-\,(\omega-\epsilon_k^+)+2\,i\,\gamma_{\mathrm{R}k}}{4\,\gamma_{\mathrm{R}k}\{(\omega-\epsilon_k^+)^2+4\,(\gamma_{\mathrm{R}k})^2\}}\,\bar{n}(\omega_{\mathrm{R}k})\{\bar{n}(\omega_{\mathrm{R}k})+1\}(\cosh^2 2\theta_k+1), \qquad \text{(C.3b)}
\end{aligned}
$$

$$
\begin{aligned}
X^{\alpha}_{k2}(\omega) &= \langle 1_{\mathrm{S}}|\, \alpha_k \,|D^{(2)}_{k2}[\omega]\rangle/(2\,(\omega+\epsilon^-_k)) = X^{\alpha}_{k2}(\omega)' + i\, X^{\alpha}_{k2}(\omega)'', \\
&= \frac{i\,(g_1^2/4)\,S\sinh 2\theta_k}{\{-i\,(\omega-\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}\{-i\,(-\epsilon^-_k-\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}} + \frac{-\,i\,(g_1^2/4)\,S\sinh 2\theta_k}{\{-i\,(\omega+\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}\{-i\,(-\epsilon^-_k+\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}} \\
&\quad + g_2^2\,\frac{i\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}\sinh 2\theta_k\cosh 2\theta_k}{2\,\{-i\,(\omega-\epsilon^+_k)+2\gamma_{\mathrm{R}k}\}\{-i\,(-\epsilon^+_k-\epsilon^-_k)+2\gamma_{\mathrm{R}k}\}} \\
&\quad - g_2^2\,\frac{i\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}\sinh 2\theta_k\cosh 2\theta_k}{4\,\gamma_{\mathrm{R}k}\,\{-i\,(\omega+\epsilon^-_k)+2\gamma_{\mathrm{R}k}\,\}}, \qquad \text{(C.4a)} \\
&= g_1^2\,S\,\frac{-\,\gamma_{\mathrm{R}k}\,(\omega-\epsilon^-_k-2\,\omega_{\mathrm{R}k})+i\,\{(\gamma_{\mathrm{R}k})^2+(\omega-\omega_{\mathrm{R}k})(\epsilon^-_k+\omega_{\mathrm{R}k})\}}{4\,\{(\omega-\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}\{(\epsilon^-_k+\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}}\sinh 2\theta_k \\
&\quad + g_1^2\,S\,\frac{\gamma_{\mathrm{R}k}\,(\omega-\epsilon^-_k+2\,\omega_{\mathrm{R}k})-i\,\{(\gamma_{\mathrm{R}k})^2+(\omega+\omega_{\mathrm{R}k})(\epsilon^-_k-\omega_{\mathrm{R}k})\}}{4\,\{(\omega+\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}\{(\epsilon^-_k-\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}}\sinh 2\theta_k \\
&\quad + g_2^2\,\frac{-\,2\,\gamma_{\mathrm{R}k}\,(\omega-2\,\epsilon^+_k-\epsilon^-_k)+i\,\{4\,(\gamma_{\mathrm{R}k})^2+(\omega-\epsilon^+_k)(\epsilon^+_k+\epsilon^-_k)\}}{2\,\{(\omega-\epsilon^+_k)^2+4\,(\gamma_{\mathrm{R}k})^2\}\{(\epsilon^+_k+\epsilon^-_k)^2+4\,(\gamma_{\mathrm{R}k})^2\}}\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}\sinh 2\theta_k\cosh 2\theta_k \\
&\quad + g_2^2\,\frac{(\omega+\epsilon^-_k)-2\,i\,\gamma_{\mathrm{R}k}}{4\,\gamma_{\mathrm{R}k}\,\{(\omega+\epsilon^-_k)^2+4\,(\gamma_{\mathrm{R}k})^2\}}\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}\sinh 2\theta_k\cosh 2\theta_k\,, \qquad \text{(C.4b)}
\end{aligned}
$$

$$
\begin{aligned}
X^{\beta}_{k1}(\omega) &= \langle 1_{\mathrm{S}}|\, \beta^\dagger_k \,|D^{(2)}_{k1}[\omega]\rangle/(2\,(\omega-\epsilon^+_k)) = X^{\beta}_{k1}(\omega)' + i\, X^{\beta}_{k2}(\omega)'', \\
&= \frac{i\,g_1^2\,S\sinh 2\theta_k}{4\,\{-i\,(\omega-\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}\{-i\,(\epsilon^+_k-\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}} + \frac{-\,i\,g_1^2\,S\sinh 2\theta_k}{4\,\{-i\,(\omega+\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}\{-i\,(\epsilon^+_k+\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}} \\
&\quad - g_2^2\,\frac{i\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}\sinh 2\theta_k\cosh 2\theta_k}{2\,\{-i\,(\omega+\epsilon^-_k)+2\gamma_{\mathrm{R}k}\}\{-i\,(\epsilon^+_k+\epsilon^-_k)+2\gamma_{\mathrm{R}k}\}} \\
&\quad + g_2^2\,\frac{i\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}}{4\,\gamma_{\mathrm{R}k}\,\{-i\,(\omega-\epsilon^+_k)+2\gamma_{\mathrm{R}k}\}}\sinh 2\theta_k\cosh 2\theta_k\,, \qquad \text{(C.5a)} \\
&= g_1^2\,S\,\frac{-\,\gamma_{\mathrm{R}k}\,(\omega+\epsilon^+_k-2\,\omega_{\mathrm{R}k})+i\,\{(\gamma_{\mathrm{R}k})^2-(\omega-\omega_{\mathrm{R}k})(\epsilon^+_k-\omega_{\mathrm{R}k})\}}{4\,\{(\omega-\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}\{(\epsilon^+_k-\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}}\sinh 2\theta_k \\
&\quad + g_1^2\,S\,\frac{\gamma_{\mathrm{R}k}\,(\omega+\epsilon^+_k+2\,\omega_{\mathrm{R}k})-i\,\{(\gamma_{\mathrm{R}k})^2-(\omega+\omega_{\mathrm{R}k})(\epsilon^+_k+\omega_{\mathrm{R}k})\}}{4\,\{(\omega+\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}\{(\epsilon^+_k+\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}}\sinh 2\theta_k \\
&\quad + g_2^2\,\frac{2\,\gamma_{\mathrm{R}k}\,(\omega+\epsilon^+_k+2\,\epsilon^-_k)-i\,\{4\,(\gamma_{\mathrm{R}k})^2-(\omega+\epsilon^-_k)(\epsilon^+_k+\epsilon^-_k)\}}{2\,\{(\omega+\epsilon^-_k)^2+4\,(\gamma_{\mathrm{R}k})^2\}\{(\epsilon^+_k+\epsilon^-_k)^2+4\,(\gamma_{\mathrm{R}k})^2\}}\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}\sinh 2\theta_k\cosh 2\theta_k \\
&\quad + g_2^2\,\frac{-\,(\omega-\epsilon^+_k)+2\,i\,\gamma_{\mathrm{R}k}}{4\,\gamma_{\mathrm{R}k}\,\{(\omega-\epsilon^+_k)^2+4\,(\gamma_{\mathrm{R}k})^2\}}\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}\sinh 2\theta_k\cosh 2\theta_k\,, \qquad \text{(C.5b)}
\end{aligned}
$$

$$
\begin{aligned}
X^{\beta}_{k2}(\omega) &= \langle 1_{\mathrm{S}}|\, \beta^\dagger_k \,|D^{(2)}_{k2}[\omega]\rangle/(2\,(\omega+\epsilon^-_k)) = X^{\beta}_{k2}(\omega)' + i\, X^{\beta}_{k2}(\omega)'', \\
&= g_1^2\,\frac{i\,S(\cosh 2\theta_k-1)}{4\,\{-i\,(\omega-\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}\{-i\,(-\epsilon^-_k-\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}} \\
&\quad + g_1^2\,\frac{-\,i\,S(\cosh 2\theta_k+1)}{4\,\{-i\,(\omega+\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}\{-i\,(-\epsilon^-_k+\omega_{\mathrm{R}k})+\gamma_{\mathrm{R}k}\}} - g_2^2\,\frac{i\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}(\cosh^2 2\theta_k+1)}{4\,\gamma_{\mathrm{R}k}\{-i\,(\omega+\epsilon^-_k)+2\gamma_{\mathrm{R}k}\}} \\
&\quad + g_2^2\,\frac{i\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}\sinh^2 2\theta_k}{2\,\{-i\,(\omega-\epsilon^+_k)+2\gamma_{\mathrm{R}k}\}\{-i\,(-\epsilon^+_k-\epsilon^-_k)+2\gamma_{\mathrm{R}k}\}}, \qquad \text{(C.6a)} \\
&= g_1^2\,S\,\frac{-\,\gamma_{\mathrm{R}k}\,(\omega-\epsilon^-_k-2\,\omega_{\mathrm{R}k})+i\,\{(\gamma_{\mathrm{R}k})^2+(\omega-\omega_{\mathrm{R}k})(\epsilon^-_k+\omega_{\mathrm{R}k})\}}{4\,\{(\omega-\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}\{(\epsilon^-_k+\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}}(\cosh 2\theta_k-1) \\
&\quad + g_1^2\,S\,\frac{\gamma_{\mathrm{R}k}\,(\omega-\epsilon^-_k+2\,\omega_{\mathrm{R}k})-i\,\{(\gamma_{\mathrm{R}k})^2+(\omega+\omega_{\mathrm{R}k})(\epsilon^-_k-\omega_{\mathrm{R}k})\}}{4\,\{(\omega+\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}\{(\epsilon^-_k-\omega_{\mathrm{R}k})^2+(\gamma_{\mathrm{R}k})^2\}}(\cosh 2\theta_k+1) \\
&\quad + g_2^2\,\frac{-\,2\,\gamma_{\mathrm{R}k}\,(\omega-2\,\epsilon^+_k-\epsilon^-_k)+i\,\{4\,(\gamma_{\mathrm{R}k})^2+(\omega-\epsilon^+_k)(\epsilon^+_k+\epsilon^-_k)\}}{2\,\{(\omega-\epsilon^+_k)^2+4\,(\gamma_{\mathrm{R}k})^2\}\{(\epsilon^+_k+\epsilon^-_k)^2+4\,(\gamma_{\mathrm{R}k})^2\}}\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}\sinh^2 2\theta_k \\
&\quad + g_2^2\,\frac{(\omega+\epsilon^-_k)-2\,i\,\gamma_{\mathrm{R}k}}{4\,\gamma_{\mathrm{R}k}\{(\omega+\epsilon^-_k)^2+4\,(\gamma_{\mathrm{R}k})^2\}}\,\bar n(\omega_{\mathrm{R}k})\{\bar n(\omega_{\mathrm{R}k})+1\}(\cosh^2 2\theta_k+1). \qquad \text{(C.6b)}
\end{aligned}
$$

D Derivation of forms of $n_k^\alpha(0)$ and $n_k^\beta(0)$

In this Appendix, we consider the case that the anti-ferromagnetic spin system and phonon reservoir are in the thermal equilibrium state at the initial time $t=0$, i.e., $\rho_{\mathrm{T}}(0)=\rho_{\mathrm{TE}}$, and derive forms of $n_k^\alpha(0)$ $[=\langle 1_{\mathrm{S}}|\alpha_k^\dagger\alpha_k|\rho_0\rangle]$ and $n_k^\beta(0)$ $[=\langle 1_{\mathrm{S}}|\beta_k^\dagger\beta_k|\rho_0\rangle]$ up to the second order in powers of the spin-phonon interaction in the lowest spin-wave approximation. The thermal state $|\rho_0\rangle$ $[=|\rho(0)\rangle=\langle 1_{\mathrm{R}}|\rho_{\mathrm{T}}(0)\rangle=\langle 1_{\mathrm{R}}|\rho_{\mathrm{TE}}\rangle]$ can be expanded in powers of the spin-phonon interaction, as [36]

$$|\rho_0\rangle = |\rho_{\mathrm{S}}\rangle + |\rho_0^{(2)}\rangle + \cdots\,, \tag{D.1}$$

with ρ_{S} given by (A.4), where $|\rho_0^{(2)}\rangle$ is the second-order part of $|\rho_0\rangle$ $[=\langle 1_{\mathrm{R}}|\rho_{\mathrm{TE}}\rangle]$ in powers of the spin-phonon interaction and is given by

$$\begin{aligned}|\rho_0^{(2)}\rangle = \int_0^\beta d\beta_1 \int_0^{\beta_1} d\beta_2\, \langle 1_{\mathrm{R}}\big|\big\{\mathcal{H}_{\mathrm{SR}}(-i\hbar\beta_1)\mathcal{H}_{\mathrm{SR}}(-i\hbar\beta_2)\\ - \langle 1|\,\mathcal{H}_{\mathrm{SR}}(-i\hbar\beta_1)\mathcal{H}_{\mathrm{SR}}(-i\hbar\beta_2)|\rho_{\mathrm{R}}\rangle|\rho_{\mathrm{S}}\rangle\big\}\big|\rho_{\mathrm{R}}\rangle|\rho_{\mathrm{S}}\rangle.\end{aligned} \tag{D.2}$$

The above form for $|\rho_0^{(2)}\rangle$ can be expressed with time-integrals alone by transforming inverse-temperature-integrals into time-integrals, as done in Ref. [36], as

$$|\rho_0^{(2)}\rangle = -\int_0^\infty d\tau_1 \int_0^{\tau_1} d\tau_2\, \langle 1_{\mathrm{R}}|\,\hat{\mathcal{H}}_{\mathrm{SR}}(-\tau_2)\,\hat{\mathcal{H}}_{\mathrm{SR}}(-\tau_1)\,|\rho_{\mathrm{R}}\rangle|\rho_{\mathrm{S}}\rangle \exp(-\mu\,\tau_1)\,\big|_{\mu\to+0}\,. \tag{D.3}$$

Here, $\mathcal{H}_{\mathrm{SR}}(t)$ and $\hat{\mathcal{H}}_{\mathrm{SR}}(t)$ are defined by $\mathcal{H}_{\mathrm{SR}}(t)=\exp(i\mathcal{H}_0 t/\hbar)\mathcal{H}_{\mathrm{SR}}\exp(-i\mathcal{H}_0 t/\hbar)$ and $\hat{\mathcal{H}}_{\mathrm{SR}}(t)=\exp(i\hat{\mathcal{H}}_0 t)\mathcal{H}_{\mathrm{SR}}\exp(-i\hat{\mathcal{H}}_0 t)$, where $\mathcal{H}_0=\mathcal{H}_{\mathrm{S}}+\mathcal{H}_{\mathrm{R}}$. By substituting (2.21) into (D.3), $\langle 1_{\mathrm{S}}|\alpha_k^\dagger\alpha_k|\rho_0^{(2)}\rangle$ and $\langle 1_{\mathrm{S}}|\beta_k^\dagger\beta_k|\rho_0^{(2)}\rangle$ can be expressed as

$$\begin{aligned}
\langle 1_{\mathrm{S}}|\alpha_k^\dagger\alpha_k|\rho_0^{(2)}\rangle &= -\int_0^\infty d\tau_1 \int_0^{\tau_1} d\tau\, \langle 1_{\mathrm{S}}|\langle 1_{\mathrm{R}}|\alpha_k^\dagger\alpha_k\hat{\mathcal{H}}_{\mathrm{SR}}(-\tau)\hat{\mathcal{H}}_{\mathrm{SR}}(-\tau_1)|\rho_{\mathrm{R}}\rangle|\rho_{\mathrm{S}}\rangle \exp(-\mu\,\tau_1)\big|_{\mu\to+0}\,,\\
&= \frac{S}{2}\int_0^\infty d\tau_1 \int_0^{\tau_1} d\tau \sum_\nu |\,g_{1\nu}\,|^2 \exp(-\mu\,\tau_1)\,\big|_{\mu\to+0}\\
&\quad\times\big\{\big(\langle 1_{\mathrm{R}}|R_{k\nu}^{a\dagger}(\tau)R_{k\nu}^{a}|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\alpha_k\alpha_k^\dagger|\rho_{\mathrm{S}}\rangle - \langle 1_{\mathrm{R}}|R_{k\nu}^{a}R_{k\nu}^{a\dagger}(\tau)|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\alpha_k^\dagger\alpha_k|\rho_{\mathrm{S}}\rangle\big)\cosh^2\theta_k \exp(-i\,\epsilon_k^+\,\tau)\\
&\quad-\big(\langle 1_{\mathrm{R}}|R_{k\nu}^{a}(\tau)R_{k\nu}^{a\dagger}|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\alpha_k^\dagger\alpha_k|\rho_{\mathrm{S}}\rangle - \langle 1_{\mathrm{R}}|R_{k\nu}^{a\dagger}R_{k\nu}^{a}(\tau)|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\alpha_k\alpha_k^\dagger|\rho_{\mathrm{S}}\rangle\big)\cosh^2\theta_k \exp(i\,\epsilon_k^+\,\tau)\\
&\quad+\big(\langle 1_{\mathrm{R}}|R_{k\nu}^{b}(\tau)R_{k\nu}^{b\dagger}|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\alpha_k\alpha_k^\dagger|\rho_{\mathrm{S}}\rangle - \langle 1_{\mathrm{R}}|R_{k\nu}^{b\dagger}R_{k\nu}^{b}(\tau)|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\alpha_k^\dagger\alpha_k|\rho_{\mathrm{S}}\rangle\big)\sinh^2\theta_k \exp(-i\,\epsilon_k^+\,\tau)\\
&\quad-\big(\langle 1_{\mathrm{R}}|R_{k\nu}^{b\dagger}(\tau)R_{k\nu}^{b}|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\alpha_k^\dagger\alpha_k|\rho_{\mathrm{S}}\rangle - \langle 1_{\mathrm{R}}|R_{k\nu}^{b}R_{k\nu}^{b\dagger}(\tau)|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\alpha_k\alpha_k^\dagger|\rho_{\mathrm{S}}\rangle\big)\sinh^2\theta_k \exp(i\,\epsilon_k^+\,\tau)\big\}\\
&\quad+\int_0^\infty d\tau_1 \int_0^{\tau_1} d\tau \sum_\nu g_{2\nu}^2 \exp(-\mu\,\tau_1)\,\big|_{\mu\to+0}\\
&\qquad\times\big\{\big\langle 1_{\mathrm{R}}\big|\Delta(R_{k\nu}^{a\dagger}(\tau)R_{k\nu}^{a}(\tau))\Delta(R_{k\nu}^{a\dagger}R_{k\nu}^{a})\big|\rho_{\mathrm{R}}\big\rangle \cosh^2\theta_k\sinh^2\theta_k\\
&\qquad\quad\times\big(\langle 1_{\mathrm{S}}|\alpha_k\beta_k\alpha_k^\dagger\beta_k^\dagger|\rho_{\mathrm{S}}\rangle \exp\{-i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} - \langle 1_{\mathrm{S}}|\alpha_k^\dagger\beta_k^\dagger\alpha_k\beta_k|\rho_{\mathrm{S}}\rangle \exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\tau\}\\
&\qquad+\big\langle 1_{\mathrm{R}}\big|\Delta(R_{k\nu}^{b\dagger}(\tau)R_{k\nu}^{b}(\tau))\Delta(R_{k\nu}^{b\dagger}R_{k\nu}^{b})\big|\rho_{\mathrm{R}}\big\rangle \cosh^2\theta_k\sinh^2\theta_k\\
&\qquad\quad\times\big(\langle 1_{\mathrm{S}}|\alpha_k\beta_k\alpha_k^\dagger\beta_k^\dagger|\rho_{\mathrm{S}}\rangle \exp\{-i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} - \langle 1_{\mathrm{S}}|\alpha_k^\dagger\beta_k^\dagger\alpha_k\beta_k|\rho_{\mathrm{S}}\rangle \exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\tau\}\\
&\qquad+\big\langle 1_{\mathrm{R}}\big|\Delta(R_{k\nu}^{a\dagger}R_{k\nu}^{a})\Delta(R_{k\nu}^{a\dagger}(\tau)R_{k\nu}^{a}(\tau))\big|\rho_{\mathrm{R}}\big\rangle \cosh^2\theta_k\sinh^2\theta_k\\
&\qquad\quad\times\big(\langle 1_{\mathrm{S}}|\alpha_k\beta_k\alpha_k^\dagger\beta_k^\dagger|\rho_{\mathrm{S}}\rangle \exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} - \langle 1_{\mathrm{S}}|\alpha_k^\dagger\beta_k^\dagger\alpha_k\beta_k|\rho_{\mathrm{S}}\rangle \exp\{-i\,(\epsilon_k^+ + \epsilon_k^-)\tau\}\\
&\qquad+\big\langle 1_{\mathrm{R}}\big|\Delta(R_{k\nu}^{b\dagger}R_{k\nu}^{b})\Delta(R_{k\nu}^{b\dagger}(\tau)R_{k\nu}^{b}(\tau))\big|\rho_{\mathrm{R}}\big\rangle \cosh^2\theta_k\sinh^2\theta_k\\
&\qquad\quad\times\big(\langle 1_{\mathrm{S}}|\alpha_k\beta_k\alpha_k^\dagger\beta_k^\dagger|\rho_{\mathrm{S}}\rangle \exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} - \langle 1_{\mathrm{S}}|\alpha_k^\dagger\beta_k^\dagger\alpha_k\beta_k|\rho_{\mathrm{S}}\rangle \exp\{-i\,(\epsilon_k^+ + \epsilon_k^-)\tau\}\big)\big\},
\end{aligned} \tag{D.4}$$

$$\begin{aligned}
\langle 1_{\mathrm{S}}|\beta_k^\dagger\beta_k|\rho_0^{(2)}\rangle &= -\int_0^\infty d\tau_1 \int_0^{\tau_1} d\tau_2 \langle 1_{\mathrm{S}}|\langle 1_{\mathrm{R}}|\beta_k^\dagger\beta_k\hat{\mathcal{H}}_{\mathrm{SR}}(-\tau_2)\hat{\mathcal{H}}_{\mathrm{SR}}(-\tau_1)|\rho_{\mathrm{R}}\rangle|\rho_{\mathrm{S}}\rangle \exp(-\mu\,\tau_1)\big|_{\mu\to+0}\,,\\
&= \frac{S}{2}\int_0^\infty d\tau_1 \int_0^{\tau_1} d\tau \sum_\nu |\,g_{1\nu}\,|^2 \exp(-\mu\,\tau_1)\,\big|_{\mu\to+0}\\
&\quad\times\big\{\big(\langle 1_{\mathrm{R}}|R_{k\nu}^{a}(\tau)R_{k\nu}^{a\dagger}|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\beta_k\beta_k^\dagger|\rho_{\mathrm{S}}\rangle - \langle 1_{\mathrm{R}}|R_{k\nu}^{a\dagger}R_{k\nu}^{a}(\tau)|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\beta_k^\dagger\beta_k|\rho_{\mathrm{S}}\rangle\big)\sinh^2\theta_k \exp(-i\,\epsilon_k^-\,\tau)\\
&\quad-\big(\langle 1_{\mathrm{R}}|R_{k\nu}^{a\dagger}(\tau)R_{k\nu}^{a}|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\beta_k^\dagger\beta_k|\rho_{\mathrm{S}}\rangle - \langle 1_{\mathrm{R}}|R_{k\nu}^{a}R_{k\nu}^{a\dagger}(\tau)|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\beta_k\beta_k^\dagger|\rho_{\mathrm{S}}\rangle\big)\sinh^2\theta_k \exp(i\,\epsilon_k^-\,\tau)\\
&\quad+\big(\langle 1_{\mathrm{R}}|R_{k\nu}^{b\dagger}(\tau)R_{k\nu}^{b}|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\beta_k\beta_k^\dagger|\rho_{\mathrm{S}}\rangle - \langle 1_{\mathrm{R}}|R_{k\nu}^{b}R_{k\nu}^{b\dagger}(\tau)|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\beta_k^\dagger\beta_k|\rho_{\mathrm{S}}\rangle\big)\cosh^2\theta_k \exp(-i\,\epsilon_k^-\,\tau)\\
&\quad-\big(\langle 1_{\mathrm{R}}|R_{k\nu}^{b}(\tau)R_{k\nu}^{b\dagger}|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\beta_k^\dagger\beta_k|\rho_{\mathrm{S}}\rangle - \langle 1_{\mathrm{R}}|R_{k\nu}^{b\dagger}R_{k\nu}^{b}(\tau)|\rho_{\mathrm{R}}\rangle\langle 1_{\mathrm{S}}|\beta_k\beta_k^\dagger|\rho_{\mathrm{S}}\rangle\big)\cosh^2\theta_k \exp(i\,\epsilon_k^-\,\tau)\big\}
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty d\tau_1 \int_0^{\tau_1} d\tau \sum_\nu g_{2\nu}^2 \exp(-\mu\,\tau_1)\,\big|_{\mu\to+0} \\
&\quad\times \big\{\big\langle 1_{\mathtt{R}}\big|\Delta(R_{k\nu}^{a\dagger}(\tau)R_{k\nu}^{a}(\tau))\Delta(R_{k\nu}^{a\dagger}R_{k\nu}^{a})\big|\rho_{\mathtt{R}}\big\rangle \cosh^2\theta_k \sinh^2\theta_k \\
&\qquad\times \big(\langle 1_{\mathtt{S}}|\alpha_k\beta_k\alpha_k^\dagger\beta_k^\dagger|\rho_{\mathtt{S}}\rangle \exp\{-i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} - \langle 1_{\mathtt{S}}|\alpha_k^\dagger\beta_k^\dagger\alpha_k\beta_k|\rho_{\mathtt{S}}\rangle \exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} \\
&\quad+ \big\langle 1_{\mathtt{R}}\big|\Delta(R_{k\nu}^{b\dagger}(\tau)R_{k\nu}^{b}(\tau))\Delta(R_{k\nu}^{b\dagger}R_{k\nu}^{b})\big|\rho_{\mathtt{R}}\big\rangle \cosh^2\theta_k \sinh^2\theta_k \\
&\qquad\times \big(\langle 1_{\mathtt{S}}|\alpha_k\beta_k\alpha_k^\dagger\beta_k^\dagger|\rho_{\mathtt{S}}\rangle \exp\{-i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} - \langle 1_{\mathtt{S}}|\alpha_k^\dagger\beta_k^\dagger\alpha_k\beta_k|\rho_{\mathtt{S}}\rangle \exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} \\
&\quad+ \big\langle 1_{\mathtt{R}}\big|\Delta(R_{k\nu}^{a\dagger}R_{k\nu}^{a})\Delta(R_{k\nu}^{a\dagger}(\tau)R_{k\nu}^{a}(\tau))\big|\rho_{\mathtt{R}}\big\rangle \cosh^2\theta_k \sinh^2\theta_k \\
&\qquad\times \big(\langle 1_{\mathtt{S}}|\alpha_k\beta_k\alpha_k^\dagger\beta_k^\dagger|\rho_{\mathtt{S}}\rangle \exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} - \langle 1_{\mathtt{S}}|\alpha_k^\dagger\beta_k^\dagger\alpha_k\beta_k|\rho_{\mathtt{S}}\rangle \exp\{-i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} \\
&\quad+ \big\langle 1_{\mathtt{R}}\big|\Delta(R_{k\nu}^{b\dagger}R_{k\nu}^{b})\Delta(R_{k\nu}^{b\dagger}(\tau)R_{k\nu}^{b}(\tau))\big|\rho_{\mathtt{R}}\big\rangle \cosh^2\theta_k \sinh^2\theta_k \\
&\qquad\times \big(\langle 1_{\mathtt{S}}|\alpha_k\beta_k\alpha_k^\dagger\beta_k^\dagger|\rho_{\mathtt{S}}\rangle \exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} - \langle 1_{\mathtt{S}}|\alpha_k^\dagger\beta_k^\dagger\alpha_k\beta_k|\rho_{\mathtt{S}}\rangle \exp\{-i\,(\epsilon_k^+ + \epsilon_k^-)\tau\}\big)\big\},
\end{aligned}
\tag{D.5}
$$

from which we can obtain the forms of $n_k^\alpha(0)$ and $n_k^\beta(0)$ up to the second order in powers of the spin-phonon interaction by using the Bose operators $R_{k\nu}$ and $R_{k\nu}^\dagger$ defined by the assumptions (2.17) and (2.24a) – (2.24c), as follows

$$
n_k^\alpha(0) = \langle 1_{\mathtt{S}}|\alpha_k^\dagger\alpha_k|\rho_0\rangle = \langle 1_{\mathtt{S}}|\alpha_k^\dagger\alpha_k|\rho_{\mathtt{S}}\rangle + \langle 1_{\mathtt{S}}|\alpha_k^\dagger\alpha_k|\rho_0^{(2)}\rangle, \tag{D.6}
$$

$$
\begin{aligned}
&= \bar{n}(\epsilon_k^+) \\
&\quad+ \frac{S}{4}\int_0^\infty d\tau \int_\tau^\infty d\tau_1 \sum_\nu |\,g_{1\nu}\,|^2 \exp(-\mu\,\tau_1)\,\big|_{\mu\to+0} \\
&\qquad\times \big\{(\cosh 2\theta_k + 1)\,\langle 1_{\mathtt{S}}|\alpha_k\alpha_k^\dagger|\rho_{\mathtt{S}}\rangle\,\{\langle 1_{\mathtt{R}}|R_{k\nu}^\dagger(\tau)R_{k\nu}|\rho_{\mathtt{R}}\rangle \exp(-\,i\,\epsilon_k^+\,\tau) + \langle 1_{\mathtt{R}}|R_{k\nu}^\dagger R_{k\nu}(\tau)|\rho_{\mathtt{R}}\rangle \exp(i\,\epsilon_k^+\,\tau)\} \\
&\qquad\quad- (\cosh 2\theta_k + 1)\,\langle 1_{\mathtt{S}}|\alpha_k^\dagger\alpha_k|\rho_{\mathtt{S}}\rangle\,\{\langle 1_{\mathtt{R}}|R_{k\nu}(\tau)R_{k\nu}^\dagger|\rho_{\mathtt{R}}\rangle \exp(i\,\epsilon_k^+\,\tau) + \langle 1_{\mathtt{R}}|R_{k\nu}R_{k\nu}^\dagger(\tau)|\rho_{\mathtt{R}}\rangle \exp(-\,i\,\epsilon_k^+\,\tau)\} \\
&\qquad\quad+ (\cosh 2\theta_k - 1)\,\langle 1_{\mathtt{S}}|\alpha_k\alpha_k^\dagger|\rho_{\mathtt{S}}\rangle\,\{\langle 1_{\mathtt{R}}|R_{k\nu}(\tau)R_{k\nu}^\dagger|\rho_{\mathtt{R}}\rangle \exp(-\,i\,\epsilon_k^+\,\tau) + \langle 1_{\mathtt{R}}|R_{k\nu}R_{k\nu}^\dagger(\tau)|\rho_{\mathtt{R}}\rangle \exp(i\,\epsilon_k^+\,\tau)\} \\
&\qquad\quad- (\cosh 2\theta_k - 1)\,\langle 1_{\mathtt{S}}|\alpha_k^\dagger\alpha_k|\rho_{\mathtt{S}}\rangle\,\{\langle 1_{\mathtt{R}}|R_{k\nu}^\dagger(\tau)R_{k\nu}|\rho_{\mathtt{R}}\rangle \exp(i\,\epsilon_k^+\,\tau) + \langle 1_{\mathtt{R}}|R_{k\nu}^\dagger R_{k\nu}(\tau)|\rho_{\mathtt{R}}\rangle \exp(-\,i\,\epsilon_k^+\,\tau)\}\big\} \\
&\quad+ \frac{1}{2}\int_0^\infty d\tau \int_\tau^\infty d\tau_1 \sum_\nu g_{2\nu}^2 \sinh^2 2\theta_k \exp(-\mu\,\tau_1)\big|_{\mu\to+0} \\
&\qquad\times \big\{\big\langle 1_{\mathtt{R}}\big|\Delta(R_{k\nu}^\dagger(\tau)R_{k\nu}(\tau))\Delta(R_{k\nu}^\dagger R_{k\nu})\big|\rho_{\mathtt{R}}\big\rangle \\
&\qquad\quad\times \big((\bar{n}(\epsilon_k^+)+1)(\bar{n}(\epsilon_k^-)+1)\exp\{-i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} - \bar{n}(\epsilon_k^+)\,\bar{n}(\epsilon_k^-)\exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} \\
&\qquad+ \big\langle 1_{\mathtt{R}}\big|\Delta(R_{k\nu}^\dagger R_{k\nu})\Delta(R_{k\nu}^\dagger(\tau)R_{k\nu}(\tau))\big|\rho_{\mathtt{R}}\big\rangle \\
&\qquad\quad\times \big((\bar{n}(\epsilon_k^+)+1)(\bar{n}(\epsilon_k^-)+1)\exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} - \bar{n}(\epsilon_k^+)\,\bar{n}(\epsilon_k^-)\exp\{-\,i\,(\epsilon_k^+ + \epsilon_k^-)\tau\}\big)\big\},
\end{aligned}
\tag{D.7}
$$

$$
\begin{aligned}
&= \bar{n}(\epsilon_k^+) - \frac{S}{2}\int_0^\infty d\tau\cdot\tau \sum_\nu |\,g_{1\nu}\,|^2 \exp(-\mu\,\tau)\big|_{\mu\to+0} \\
&\qquad\times \big\{(\cosh 2\theta_k + 1)(\bar{n}(\epsilon_k^+)+1)\,\mathrm{Re}\,\langle 1_{\mathtt{R}}|R_{k\nu}^\dagger(\tau)R_{k\nu}|\rho_{\mathtt{R}}\rangle \exp(-\,i\,\epsilon_k^+\,\tau) \\
&\qquad\quad- (\cosh 2\theta_k + 1)\,\bar{n}(\epsilon_k^+)\,\mathrm{Re}\,\langle 1_{\mathtt{R}}|R_{k\nu}(\tau)R_{k\nu}^\dagger|\rho_{\mathtt{R}}\rangle \exp(i\,\epsilon_k^+\,\tau) \\
&\qquad\quad+ (\cosh 2\theta_k - 1)(\bar{n}(\epsilon_k^+)+1)\,\mathrm{Re}\,\langle 1_{\mathtt{R}}|R_{k\nu}(\tau)R_{k\nu}^\dagger|\rho_{\mathtt{R}}\rangle \exp(-\,i\,\epsilon_k^+\,\tau) \\
&\qquad\quad- (\cosh 2\theta_k - 1)\,\bar{n}(\epsilon_k^+)\,\mathrm{Re}\,\langle 1_{\mathtt{R}}|R_{k\nu}^\dagger(\tau)R_{k\nu}|\rho_{\mathtt{R}}\rangle \exp(i\,\epsilon_k^+\,\tau)\big\} \\
&\qquad- \int_0^\infty d\tau\cdot\tau \sinh^2 2\theta_k\big\{(\bar{n}(\epsilon_k^+)+1)(\bar{n}(\epsilon_k^-)+1) - \bar{n}(\epsilon_k^+)\,\bar{n}(\epsilon_k^-)\big\}\exp(-\mu\,\tau)\big|_{\mu\to+0} \\
&\qquad\quad\times \mathrm{Re}\sum_\nu g_{2\nu}^2\big\langle 1_{\mathtt{R}}\big|\Delta(R_{k\nu}^\dagger(\tau)R_{k\nu}(\tau))\Delta(R_{k\nu}^\dagger R_{k\nu})\big|\rho_{\mathtt{R}}\big\rangle \exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\tau\},
\end{aligned}
\tag{D.8}
$$

$$
\begin{aligned}
&= \bar{n}(\epsilon_k^+) - S\,(\cosh 2\theta_k + 1)\big\{(\bar{n}(\epsilon_k^+)+1)\,\mathrm{Re}\,i\,\frac{\partial}{\partial\epsilon_k^+}\,\phi_k^{+-}(\epsilon_k^+) + \bar{n}(\epsilon_k^+)\,\mathrm{Re}\,i\,\frac{\partial}{\partial\epsilon_k^+}\,\phi_k^{-+}(\epsilon_k^+)\big\} \\
&\qquad- S\,(\cosh 2\theta_k - 1)\big\{(\bar{n}(\epsilon_k^+)+1)\,\mathrm{Re}\,i\,\frac{\partial}{\partial\epsilon_k^+}\,\phi_k^{-+}(-\,\epsilon_k^+) + \bar{n}(\epsilon_k^+)\,\mathrm{Re}\,i\,\frac{\partial}{\partial\epsilon_k^+}\,\phi_k^{+-}(-\,\epsilon_k^+)\big\} \\
&\qquad+ \sinh^2 2\theta_k\big\{\bar{n}(\epsilon_k^+) + \bar{n}(\epsilon_k^-) + 1\big\}\,\mathrm{Re}\,i\,\frac{\partial}{\partial(\epsilon_k^+ + \epsilon_k^-)}\,\phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-),
\end{aligned}
\tag{D.9}
$$

$$
\begin{aligned}
&= \bar n(\epsilon_k^+) - g_1^2\, S\,(\cosh 2\theta_k + 1)(\bar n(\epsilon_k^+)+1)\,\mathrm{Re}\,\frac{i}{2}\cdot\frac{\partial}{\partial\epsilon_k^+}\Big(\frac{\bar n(\omega_{\mathtt{R}k})}{i\,(\epsilon_k^+ - \omega_{\mathtt{R}k}) + \gamma_{\mathtt{R}k}}\\
&\qquad - g_1^2\, S\,(\cosh 2\theta_k + 1)\,\bar n(\epsilon_k^+)\,\mathrm{Re}\,\frac{i}{2}\cdot\frac{\partial}{\partial\epsilon_k^+}\Big(\frac{\bar n(\omega_{\mathtt{R}k})+1}{-\,i\,(\epsilon_k^+ - \omega_{\mathtt{R}k}) + \gamma_{\mathtt{R}k}}\\
&\qquad - g_1^2\, S\,(\cosh 2\theta_k - 1)(\bar n(\epsilon_k^+)+1)\,\mathrm{Re}\,\frac{i}{2}\cdot\frac{\partial}{\partial\epsilon_k^+}\Big(\frac{\bar n(\omega_{\mathtt{R}k})+1}{i\,(\epsilon_k^+ + \omega_{\mathtt{R}k}) + \gamma_{\mathtt{R}k}}\\
&\qquad - g_1^2\, S\,(\cosh 2\theta_k - 1)\,\bar n(\epsilon_k^+)\,\mathrm{Re}\,\frac{i}{2}\cdot\frac{\partial}{\partial\epsilon_k^+}\Big(\frac{\bar n(\omega_{\mathtt{R}k})}{-\,i\,(\epsilon_k^+ + \omega_{\mathtt{R}k}) + \gamma_{\mathtt{R}k}}\\
&\quad + g_2^2\sinh^2 2\theta_k\big\{\bar n(\epsilon_k^+)+\bar n(\epsilon_k^-)+1\big\}\,\mathrm{Re}\,i\,\frac{\partial}{\partial(\epsilon_k^+ + \epsilon_k^-)}\Big(\frac{\bar n(\omega_{\mathtt{R}k})\,(\bar n(\omega_{\mathtt{R}k})+1)}{-\,i\,(\epsilon_k^+ + \epsilon_k^-) + 2\,\gamma_{\mathtt{R}k}}\quad , \qquad\qquad \text{(D.10)}\\
&= \bar n(\epsilon_k^+) + g_1^2\, S\,(\cosh 2\theta_k + 1)\,\{\bar n(\omega_{\mathtt{R}k}) - \bar n(\epsilon_k^+)\}\,\frac{(\epsilon_k^+ - \omega_{\mathtt{R}k})^2 - (\gamma_{\mathtt{R}k})^2}{2\,\{(\epsilon_k^+ - \omega_{\mathtt{R}k})^2 + (\gamma_{\mathtt{R}k})^2\}^2}\\
&\qquad + g_1^2\, S\,(\cosh 2\theta_k - 1)\,\{\bar n(\epsilon_k^+) + \bar n(\omega_{\mathtt{R}k}) + 1\}\,\frac{(\epsilon_k^+ + \omega_{\mathtt{R}k})^2 - (\gamma_{\mathtt{R}k})^2}{2\,\{(\epsilon_k^+ + \omega_{\mathtt{R}k})^2 + (\gamma_{\mathtt{R}k})^2\}^2}\\
&\quad + g_2^2\sinh^2 2\theta_k\,\{\bar n(\epsilon_k^+)+\bar n(\epsilon_k^-)+1\}\,\bar n(\omega_{\mathtt{R}k})\,\{\bar n(\omega_{\mathtt{R}k})+1\}\,\frac{(\epsilon_k^+ + \epsilon_k^-)^2 - 4\,(\gamma_{\mathtt{R}k})^2}{\{(\epsilon_k^+ + \epsilon_k^-)^2 + 4\,(\gamma_{\mathtt{R}k})^2\}^2}, \qquad\qquad \text{(D.11)}
\end{aligned}
$$

$$
\begin{aligned}
&n_k^\beta(0) = \langle 1_{\mathtt{S}}|\beta_k^\dagger\beta_k|\rho_0\rangle = \langle 1_{\mathtt{S}}|\beta_k^\dagger\beta_k|\rho_{\mathtt{S}}\rangle + \langle 1_{\mathtt{S}}|\beta_k^\dagger\beta_k|\rho_0^{(2)}\rangle, \qquad\qquad \text{(D.12)}\\
&= \bar n(\epsilon_k^-)\\
&\quad + \frac{S}{4}\int_0^\infty d\tau\int_\tau^\infty d\tau_1\sum_\nu |\,g_{1\nu}\,|^2\exp(-\,\mu\,\tau_1)\,\big|_{\mu\to+0}\\
&\qquad \times\big\{(\cosh 2\theta_k - 1)\,\langle 1_{\mathtt{S}}|\beta_k\beta_k^\dagger|\rho_0\rangle\,\{\langle 1_{\mathtt{R}}|R_{k\nu}(\tau)R_{k\nu}^\dagger|\rho_{\mathtt{R}}\rangle\exp(-\,i\,\epsilon_k^-\,\tau) + \langle 1_{\mathtt{R}}|R_{k\nu}R_{k\nu}^\dagger(\tau)|\rho_{\mathtt{R}}\rangle\exp(i\,\epsilon_k^-\,\tau)\}\\
&\qquad\quad - (\cosh 2\theta_k - 1)\,\langle 1_{\mathtt{S}}|\beta_k^\dagger\beta_k|\rho_0\rangle\,\{\langle 1_{\mathtt{R}}|R_{k\nu}^\dagger(\tau)R_{k\nu}|\rho_{\mathtt{R}}\rangle\exp(i\,\epsilon_k^-\,\tau) + \langle 1_{\mathtt{R}}|R_{k\nu}^\dagger R_{k\nu}(\tau)|\rho_{\mathtt{R}}\rangle\exp(-\,i\,\epsilon_k^-\,\tau)\}\\
&\qquad\quad + (\cosh 2\theta_k + 1)\,\langle 1_{\mathtt{S}}|\beta_k\beta_k^\dagger|\rho_0\rangle\,\{\langle 1_{\mathtt{R}}|R_{k\nu}^\dagger(\tau)R_{k\nu}|\rho_{\mathtt{R}}\rangle\exp(-\,i\,\epsilon_k^-\,\tau) + \langle 1_{\mathtt{R}}|R_{k\nu}^\dagger R_{k\nu}(\tau)|\rho_{\mathtt{R}}\rangle\exp(i\,\epsilon_k^-\,\tau)\}\\
&\qquad\quad - (\cosh 2\theta_k + 1)\,\langle 1_{\mathtt{S}}|\beta_k^\dagger\beta_k|\rho_0\rangle\,\{\langle 1_{\mathtt{R}}|R_{k\nu}(\tau)R_{k\nu}^\dagger|\rho_{\mathtt{R}}\rangle\exp(i\,\epsilon_k^-\,\tau) + \langle 1_{\mathtt{R}}|R_{k\nu}R_{k\nu}^\dagger(\tau)|\rho_{\mathtt{R}}\rangle\exp(-\,i\,\epsilon_k^-\,\tau)\}\big\}\\
&\quad + \frac{1}{2}\int_0^\infty d\tau\int_\tau^\infty d\tau_1\,\sum_\nu g_{2\nu}^2\sinh^2 2\theta_k\exp(-\,\mu\,\tau_1)\big|_{\mu\to+0}\\
&\quad \times\big\{\big\langle 1_{\mathtt{R}}\big|\Delta(R_{k\nu}^\dagger(\tau)R_{k\nu}(\tau))\Delta(R_{k\nu}^\dagger R_{k\nu})\big|\rho_{\mathtt{R}}\big\rangle\\
&\qquad \times\big((\bar n(\epsilon_k^+)+1)(\bar n(\epsilon_k^-)+1)\exp\{-i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} - \bar n(\epsilon_k^+)\,\bar n(\epsilon_k^-)\exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\tau\}\\
&\qquad + \big\langle 1_{\mathtt{R}}\big|\Delta(R_{k\nu}^\dagger R_{k\nu})\Delta(R_{k\nu}^\dagger(\tau)R_{k\nu}(\tau))\big|\rho_{\mathtt{R}}\big\rangle\\
&\qquad \times\big((\bar n(\epsilon_k^+)+1)(\bar n(\epsilon_k^-)+1)\exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\tau\} - \bar n(\epsilon_k^+)\,\bar n(\epsilon_k^-)\exp\{-\,i\,(\epsilon_k^+ + \epsilon_k^-)\tau\}\big)\big\}, \qquad\qquad \text{(D.13)}\\
&= \bar n(\epsilon_k^-) - \frac{S}{2}\int_0^\infty d\tau\cdot\tau\sum_\nu |\,g_{1\nu}\,|^2\exp(-\,\mu\,\tau)\big|_{\mu\to+0}\\
&\qquad\quad \times\big\{(\cosh 2\theta_k - 1)(\bar n(\epsilon_k^-)+1)\,\mathrm{Re}\,\langle 1_{\mathtt{R}}|R_{k\nu}(\tau)R_{k\nu}^\dagger|\rho_{\mathtt{R}}\rangle\exp(-\,i\,\epsilon_k^-\,\tau)\\
&\qquad\qquad - (\cosh 2\theta_k - 1)\,\bar n(\epsilon_k^-)\,\mathrm{Re}\,\langle 1_{\mathtt{R}}|R_{k\nu}^\dagger(\tau)R_{k\nu}|\rho_{\mathtt{R}}\rangle\exp(i\,\epsilon_k^-\,\tau)\\
&\qquad\qquad + (\cosh 2\theta_k + 1)(\bar n(\epsilon_k^-)+1)\,\mathrm{Re}\,\langle 1_{\mathtt{R}}|R_{k\nu}^\dagger(\tau)R_{k\nu}|\rho_{\mathtt{R}}\rangle\exp(-\,i\,\epsilon_k^-\,\tau)\\
&\qquad\qquad - (\cosh 2\theta_k + 1)\,\bar n(\epsilon_k^-)\,\mathrm{Re}\,\langle 1_{\mathtt{R}}|R_{k\nu}(\tau)R_{k\nu}^\dagger|\rho_{\mathtt{R}}\rangle\exp(i\,\epsilon_k^-\,\tau)\big\}\\
&\qquad - \int_0^\infty d\tau\cdot\tau\sinh^2 2\theta_k\big\{(\bar n(\epsilon_k^+)+1)(\bar n(\epsilon_k^-)+1) - \bar n(\epsilon_k^+)\,\bar n(\epsilon_k^-)\big\}\exp(-\,\mu\,\tau)\big|_{\mu\to+0}\\
&\qquad\qquad \times\mathrm{Re}\sum_\nu g_{2\nu}^2\big\langle 1_{\mathtt{R}}\big|\Delta(R_{k\nu}^\dagger(\tau)R_{k\nu}(\tau))\Delta(R_{k\nu}^\dagger R_{k\nu})\big|\rho_{\mathtt{R}}\big\rangle\exp\{i\,(\epsilon_k^+ + \epsilon_k^-)\tau\}, \qquad\qquad \text{(D.14)}\\
&= \bar n(\epsilon_k^-) - S\,(\cosh 2\theta_k - 1)\big\{(\bar n(\epsilon_k^-)+1)\,\mathrm{Re}\,i\,\frac{\partial}{\partial\epsilon_k^-}\,\phi_k^{-+}(-\,\epsilon_k^-) + \bar n(\epsilon_k^-)\,\mathrm{Re}\,i\,\frac{\partial}{\partial\epsilon_k^-}\,\phi_k^{+-}(-\,\epsilon_k^-)\big\}\\
&\qquad - S\,(\cosh 2\theta_k + 1)\big\{(\bar n(\epsilon_k^-)+1)\,\mathrm{Re}\,i\,\frac{\partial}{\partial\epsilon_k^-}\,\phi_k^{+-}(\epsilon_k^-) + \bar n(\epsilon_k^-)\,\mathrm{Re}\,i\,\frac{\partial}{\partial\epsilon_k^-}\,\phi_k^{-+}(\epsilon_k^-)\big\}\\
&\qquad + \sinh^2 2\theta_k\big\{\bar n(\epsilon_k^+)+\bar n(\epsilon_k^-)+1\big\}\,\mathrm{Re}\,i\,\frac{\partial}{\partial(\epsilon_k^+ + \epsilon_k^-)}\phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-), \qquad\qquad \text{(D.15)}
\end{aligned}
$$

$$
\begin{aligned}
&= \bar{n}(\epsilon_k^-) - g_1^2\, S\,(\cosh 2\theta_k - 1)\,(\bar{n}(\epsilon_k^-) + 1)\,\mathrm{Re}\,\frac{i}{2}\cdot\frac{\partial}{\partial \epsilon_k^-}\Big(\frac{\bar{n}(\omega_{\mathrm{R}k}) + 1}{i\,(\epsilon_k^- + \omega_{\mathrm{R}k}) + \gamma_{\mathrm{R}k}} \\
&\qquad - g_1^2\, S\,(\cosh 2\theta_k - 1)\,\bar{n}(\epsilon_k^-)\,\mathrm{Re}\,\frac{i}{2}\cdot\frac{\partial}{\partial \epsilon_k^-}\Big(\frac{\bar{n}(\omega_{\mathrm{R}k})}{-\,i\,(\epsilon_k^- + \omega_{\mathrm{R}k}) + \gamma_{\mathrm{R}k}} \\
&\qquad - g_1^2\, S\,(\cosh 2\theta_k + 1)\,(\bar{n}(\epsilon_k^-) + 1)\,\mathrm{Re}\,\frac{i}{2}\cdot\frac{\partial}{\partial \epsilon_k^-}\Big(\frac{\bar{n}(\omega_{\mathrm{R}k})}{i\,(\epsilon_k^- - \omega_{\mathrm{R}k}) + \gamma_{\mathrm{R}k}} \\
&\qquad - g_1^2\, S\,(\cosh 2\theta_k + 1)\,\bar{n}(\epsilon_k^-)\,\mathrm{Re}\,\frac{i}{2}\cdot\frac{\partial}{\partial \epsilon_k^-}\Big(\frac{\bar{n}(\omega_{\mathrm{R}k}) + 1}{-\,i\,(\epsilon_k^- - \omega_{\mathrm{R}k}) + \gamma_{\mathrm{R}k}} \\
&\quad + g_2^2 \sinh^2 2\theta_k \big\{\bar{n}(\epsilon_k^+) + \bar{n}(\epsilon_k^-) + 1\big\}\,\mathrm{Re}\,i\cdot\frac{\partial}{\partial(\epsilon_k^+ + \epsilon_k^-)}\Big(\frac{\bar{n}(\omega_{\mathrm{R}k})\,(\bar{n}(\omega_{\mathrm{R}k}) + 1)}{-\,i\,(\epsilon_k^+ + \epsilon_k^-) + 2\,\gamma_{\mathrm{R}k}}\ , \qquad \text{(D.16)}
\end{aligned}
$$

$$
\begin{aligned}
&= \bar{n}(\epsilon_k^-) + g_1^2\, S\,(\cosh 2\theta_k - 1)\big\{\bar{n}(\epsilon_k^-) + \bar{n}(\omega_{\mathrm{R}k}) + 1\big\}\,\frac{(\epsilon_k^- + \omega_{\mathrm{R}k})^2 - (\gamma_{\mathrm{R}k})^2}{2\,\{(\epsilon_k^- + \omega_{\mathrm{R}k})^2 + (\gamma_{\mathrm{R}k})^2\}^2} \\
&\qquad + g_1^2\, S\,(\cosh 2\theta_k + 1)\big\{\bar{n}(\omega_{\mathrm{R}k}) - \bar{n}(\epsilon_k^-)\big\}\,\frac{(\epsilon_k^- - \omega_{\mathrm{R}k})^2 - (\gamma_{\mathrm{R}k})^2}{2\,\{(\epsilon_k^- - \omega_{\mathrm{R}k})^2 + (\gamma_{\mathrm{R}k})^2\}^2} \\
&\quad + g_2^2 \sinh^2 2\theta_k\,\{\bar{n}(\epsilon_k^+) + \bar{n}(\epsilon_k^-) + 1\}\,\bar{n}(\omega_{\mathrm{R}k})\,\{\bar{n}(\omega_{\mathrm{R}k}) + 1\}\,\frac{(\epsilon_k^+ + \epsilon_k^-)^2 - 4\,(\gamma_{\mathrm{R}k})^2}{\{(\epsilon_k^+ + \epsilon_k^-)^2 + 4\,(\gamma_{\mathrm{R}k})^2\}^2}, \qquad \text{(D.17)}
\end{aligned}
$$

where we have used the assumption that the phonon correlation function (2.24c) is real. Here, $\bar{n}(\epsilon)$ is given by (A.41).

E Investigation of the region valid for the lowest spin-wave approximation

In this Appendix, we investigate numerically the region valid for the lowest spin-wave approximation in the antiferromagnetic system of one-dimensional infinite spins. When the expectation values of the second terms $n_l/(4S)$ $[=a_l^\dagger a_l/(4S)]$ and $n_m/(4S)$ $[=b_m^\dagger b_m/(4S)]$ in the expansions (2.3) and (2.5) respectively, are much smaller than 1 or are smaller than about 0.01, the lowest spin-wave approximation becomes valid. In order to investigate the region valid for the lowest spin-wave approximation, we consider the expectation values $n^a(t)$ and $n^b(t)$ of the up-spin deviation number $a_l^\dagger a_l$ $[=n_l]$ and down-spin deviation number $b_m^\dagger b_m$ $[=n_m]$, which are, respectively, referred to as "the up-spin deviation number" and "the down-spin deviation number", and define $n^a(t)$ and $n^b(t)$ by

$$
n^a(t) = \frac{2}{N}\,\langle 1_{\mathrm{S}}|\sum_l a_l^\dagger a_l\,|\rho(t)\rangle = \frac{2}{N}\sum_k \langle 1_{\mathrm{S}}|\,a_k^\dagger a_k\,U(t)\,\exp_{\leftarrow}\Big\{-\,i\int_0^t d\tau\,\hat{\mathcal{H}}_{\mathrm{S}1}(\tau)\Big\}|\rho_0\rangle, \qquad \text{(E.1a)}
$$

$$
n^b(t) = \frac{2}{N}\,\langle 1_{\mathrm{S}}|\sum_m b_m^\dagger b_m\,|\rho(t)\rangle = \frac{2}{N}\sum_k \langle 1_{\mathrm{S}}|\,b_k^\dagger b_k\,U(t)\,\exp_{\leftarrow}\Big\{-\,i\int_0^t d\tau\,\hat{\mathcal{H}}_{\mathrm{S}1}(\tau)\Big\}|\rho_0\rangle, \qquad \text{(E.1b)}
$$

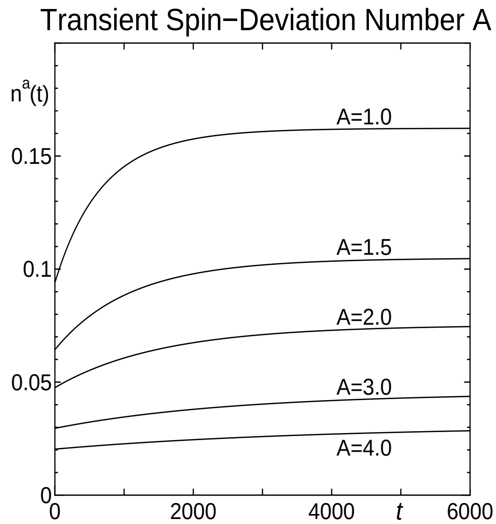
with $|\rho_0\rangle = \langle 1_{\mathrm{R}}|\rho_{\mathrm{TE}}\rangle$, where we have performed the Fourier transformations (2.7a) and (2.7b). Here, ρ_{TE} is the thermal equilibrium density operator for the spin system and phonon reservoir and is given by (A.3). In the lowest spin-wave approximation, the expectation values $n^a(t)$ and $n^b(t)$ of the up-spin deviation number and down-spin deviation number can be expressed using $n_k^\alpha(t)$ and $n_k^\beta(t)$ defined by (A.32), as

$$
n^a(t) = \frac{2}{N}\sum_k \langle 1_{\mathrm{S}}|\,a_k^\dagger a_k\,U(t)\,|\rho_0\rangle = \frac{1}{N}\sum_k \{\cosh 2\theta_k\,\big(n_k^\alpha(t) + n_k^\beta(t) + 1\ \ + n_k^\alpha(t) - n_k^\beta(t) - 1\}, \qquad \text{(E.2a)}
$$

$$
n^b(t) = \frac{2}{N}\sum_k \langle 1_{\mathrm{S}}|\,b_k^\dagger b_k\,U(t)\,|\rho_0\rangle = \frac{1}{N}\sum_k \{\cosh 2\theta_k\,\big(n_k^\alpha(t) + n_k^\beta(t) + 1\ \ - n_k^\alpha(t) + n_k^\beta(t) - 1\}, \qquad \text{(E.2b)}
$$

where we have transformed according to the transformations (2.11) and their Hermite conjugates, and have considered the axioms (A.26). The expectation values $n^a(t)$ and $n^b(t)$ of the up-spin deviation number and down-spin deviation number, given by (E.2a) and (E.2b) respectively, can be calculated by substituting (A.48a), (A.48b), (4.12a) and (4.12b) into (E.2a) and (E.2b), and by replacing the wave-number summations with the numerical integration (4.14).

In Figs. 21 and 22, the expectation values $n^a(t)$ and $n^b(t)$ of the up-spin deviation number and down-spin deviation number, respectively, are displayed varying the time t scaled by $1/J_1$ from 0 to 6000 for the cases of anisotropy energies $\hbar K$ given by $A = K/J_1 = 1.0, 1.5, 2.0, 3.0, 4.0$, and for the spin-magnitude $S = 5/2$ and the temperature T given by $k_{\mathrm{B}}T/(\hbar J_1) = 1.0$, with ζ $[=J_2/J_1]$=1.0 and ω_z/J_1=1.0. The anisotropy energy is denoted as "A" $[=K/J_1]$ in the figures. Figs. 21 and 22 show that as the time t becomes large, the expectation values $n^a(t)$ and $n^b(t)$, increase and approach to the finite values, and that as the anisotropy energy $\hbar K$ increases, the expectation values $n^a(t)$ and $n^b(t)$ decrease. Thus, the expectation values $n^a(t)$ and $n^b(t)$ given by (E.2a) and (E.2b) are the increase functions of the



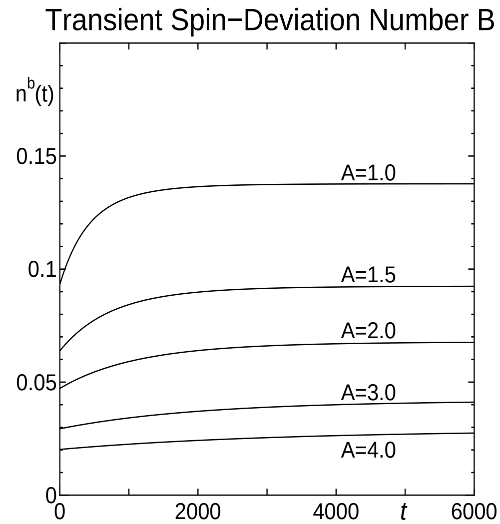


Figure 21: Up-spin-deviation number $n^a(t)$ given by (E.2a) are displayed varying the time t scaled by $1/J_1$ from 0 to 6000 for the cases of anisotropy energies $\hbar K$ given by $A = K/J_1 = 1.0, 1.5, 2.0, 3.0, 4.0$, and for the spin-magnitude $S = 5/2$ and the temperature T given by $k_{\mathrm{B}}T/(\hbar J_1) = 1.0$, with $J_2/J_1 = 1.0$ and $\omega_z/J_1 = 1.0$.

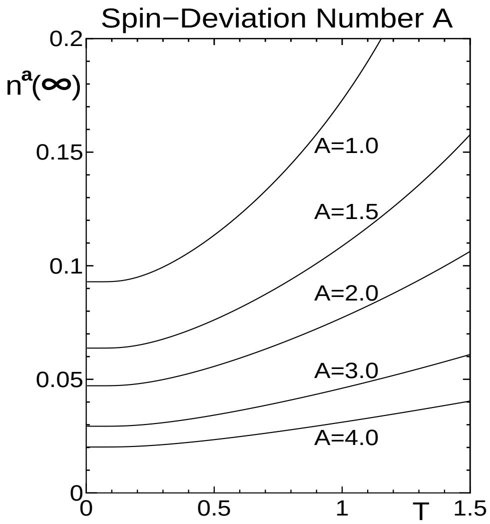
Figure 22: Down-spin-deviation number $n^b(t)$ given by (E.2b) are displayed varying the time t scaled by $1/J_1$ from 0 to 6000 for the cases of anisotropy energies $\hbar K$ given by $A = K/J_1 = 1.0, 1.5, 2.0, 3.0, 4.0$, and for the spin-magnitude $S = 5/2$ and the temperature T given by $k_{\mathrm{B}}T/(\hbar J_1) = 1.0$, with $J_2/J_1 = 1.0$ and $\omega_z/J_1 = 1.0$.

time t and the decrease functions of the anisotropy energy $\hbar K$, and approach the expectation values $n^a(\infty)$ and $n^b(\infty)$ in the infinite time limit, respectively, as time t becomes infinite $(t \to \infty)$ in no external driving magnetic field. In order to confirm the region valid for the lowest spin-wave approximation, we investigate numerically the expectation values n^a $[= n^a(\infty)]$ and n^b $[= n^b(\infty)]$ of the up-spin deviation number and down-spin deviation number in the infinite time limit $(t \to \infty)$:

$$n^a = n^a(\infty) = \frac{1}{N}\sum_k \{\cosh 2\theta_k \left(n_k^\alpha(\infty) + n_k^\beta(\infty) + 1 \right. + n_k^\alpha(\infty) - n_k^\beta(\infty) - 1\}, \tag{E.3a}$$

$$n^b = n^b(\infty) = \frac{1}{N}\sum_k \{\cosh 2\theta_k \left(n_k^\alpha(\infty) + n_k^\beta(\infty) + 1 \right. - n_k^\alpha(\infty) + n_k^\beta(\infty) - 1\}, \tag{E.3b}$$

with $n_k^\alpha(\infty)$ and $n_k^\beta(\infty)$ given by (A.49a) and (A.49b), where $n^a(\infty)$ and $n^b(\infty)$ are the expectation values in the stationary state at which the thermal equilibrium state arrives being driven by the evolution operator $U(t) = \exp\{-i\,(\hat{\mathcal{H}}_{\mathrm{S}0} + i\,C^{(2)})\,t\}$. In Figs. 23 and 24, the expectation values n^a $[= n^a(\infty)]$ and n^b $[= n^b(\infty)]$ of the up-spin deviation number and down-spin deviation number in the infinite time limit $(t \to \infty)$, respectively, are displayed varying the temperatures T scaled by $\hbar J_1/k_{\mathrm{B}}$ from 0 to 1.5 for the cases of anisotropy energies $\hbar K$ given by $A = K/J_1 = 1.0, 1.5, 2.0, 3.0, 4.0$, and for the spin-magnitude $S = 5/2$, with ζ $[= J_2/J_1]$=1.0, ω_z/J_1 =1.0. The anisotropy energy is denoted as "A" $[= K/J_1]$ in the figures. Figures 23 and 24 show that the expectation values n^a $[= n^a(\infty)]$ and n^b $[= n^b(\infty)]$ of the up-spin deviation number and down-spin deviation number, are smaller than about 0.1 in the regions of the temperature T and anisotropy energy $\hbar K$ given by $k_{\mathrm{B}}T/(\hbar J_1) \leq 1.0$ and $K/J_1 \geq 1.5$, or by $k_{\mathrm{B}}T/(\hbar J_1) \leq 1.5$ and $K/J_1 \geq 2.0$. Therefore, when $S \geq 5/2$, ζ $[= J_2/J_1] = 1.0$ and $\omega_z/J_1 = 1.0$, Figs. 23 and 24 show that $n^a/(4S)$ $[= \langle n_l \rangle/(4S)]$ and $n^b/(4S)$ $[= \langle n_m \rangle/(4S)]$, which correspond to the expectation values of the second terms in the expansions given by Eqs. (2.3) and (2.5) respectively, are smaller than about 0.01 in the regions of the temperature T and anisotropy energy $\hbar K$ given by $k_{\mathrm{B}}T/(\hbar J_1) \leq 1.0$ and $K/J_1 \geq 1.5$, or by $k_{\mathrm{B}}T/(\hbar J_1) \leq 1.5$ and $K/J_1 \geq 2.0$. In such a region, the lowest spin-wave approximation is valid. In Figs. 25 and 26, the expectation values n^a and n^b of the up-spin-deviation number and down-spin deviation-number in the infinite time limit $(t \to \infty)$, respectively, are displayed varying the anisotropy energy $\hbar K$ scaled by $\hbar J_1$ from 1.0 to 4.0 for the cases of spin-magnitudes $S = 2, 5/2, 3, 4, 5$, and for the temperature T given by $k_{\mathrm{B}}T/(\hbar J_1) = 1.0$, with ζ $[= J_2/J_1] = 1.0$, $\omega_z/J_1 = 1.0$. The anisotropy energy is denoted as "A" $[= K/J_1]$ in the figures. In the Figs. 25 and 26, we can confirm the region of the spin-magnitudes S and anisotropy energy $\hbar K$ in which $n^a/(4S)$ $[= \langle n_l \rangle/(4S)]$ and $n^b/(4S)$ $[= \langle n_m \rangle/(4S)]$, which correspond to the expectation values of the second terms in the expansions given by Eqs. (2.3) and (2.5) respectively, are smaller than about 0.01 in the region of the temperature T given by $k_{\mathrm{B}}T/(\hbar J_1) \leq 1.0$. When the temperature T is in the region given by $k_{\mathrm{B}}T/(\hbar J_1) \leq 1.0$, we can confirm the region valid for the lowest spin-wave approximation in Figs. 25 and 26.



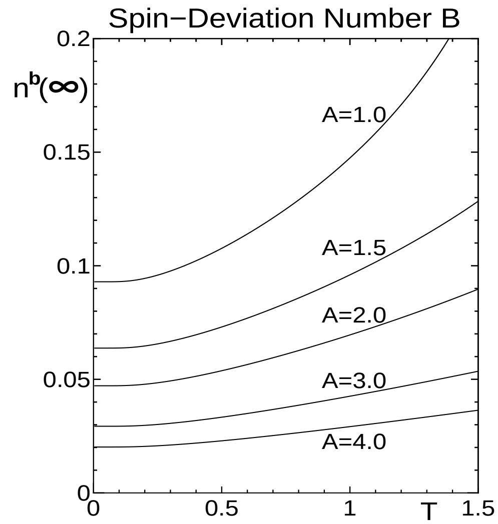
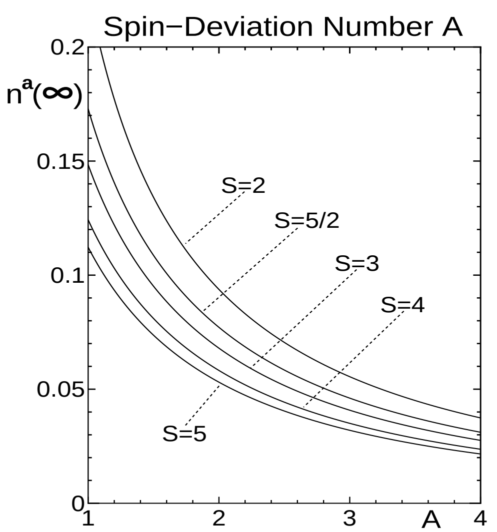


Figure 23: Up-spin-deviation number n^a $[=n^a(\infty)]$ is displayed varying the temperatures T scaled by $\hbar J_1/k_{\rm B}$ from 0 to 1.5 for the cases of anisotropy energies $\hbar K$ given by $A=K/J_1=1.0, 1.5, 2.0, 3.0, 4.0$, and for the spin-magnitude $S=5/2$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$.

Figure 24: Down-spin-deviation number n^b $[=n^b(\infty)]$ is displayed varying the temperatures T scaled by $\hbar J_1/k_{\rm B}$ from 0 to 1.5 for the cases of anisotropy energies $\hbar K$ given by $A=K/J_1=1.0, 1.5, 2.0, 3.0, 4.0$, and for the spin-magnitude $S=5/2$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$.



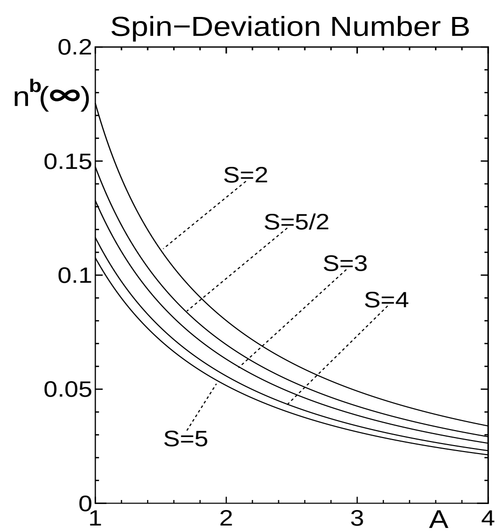


Figure 25: Up-spin-deviation number n^a $[=n^a(\infty)]$ is displayed varying the anisotropy energy $\hbar K$ scaled by $\hbar J_1$ from 1.0 to 4.0, i.e., $A=K/J_1=1.0\sim 4.0$ for the cases of spin-magnitudes $S=2, 5/2, 3, 4, 5$, and for the temperature T given by $k_{\rm B}T/(\hbar J_1)=1.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$.

Figure 26: Down-spin-deviation number n^b $[=n^b(\infty)]$ is displayed varying the anisotropy energy $\hbar K$ scaled by $\hbar J_1$ from 1.0 to 4.0, i.e., $A=K/J_1=1.0\sim 4.0$ for the cases of spin-magnitudes $S=2, 5/2, 3, 4, 5$, and for the temperature T given by $k_{\rm B}T/(\hbar J_1)=1.0$, with $J_2/J_1=1.0$ and $\omega_z/J_1=1.0$.

F The forms of $C^{(2)}$ and $|D^{(2)}_{S^-_k}[\omega]\rangle$ for $\mathcal{H}_{\mathtt{SR}}$ taken in Refs. [21, 22]

In this Appendix, the forms of the collision operator $C^{(2)}$ and the interference thermal state $|D^{(2)}_{S^-_k}[\omega]\rangle$ are given for the interaction $\mathcal{H}_{\mathtt{SR}}$ between the anti-ferromagnetic system and phonon reservoir, taken in the previous papers [21, 22], which was taken to point all of the spins to the "down" direction by the phonon-reservoir field, and thus the spin-phonon interaction $\mathcal{H}_{\mathtt{SR}}$ taken in the previous papers [21, 22] does not reflect the energy transfer between the spin system and phonon reservoir at the "down" spin sites. In the previous papers [21, 22], the spin-phonon interaction $\mathcal{H}_{\mathtt{SR}}$ was taken as

$$\begin{aligned}\mathcal{H}_{\mathtt{SR}} = &-\frac{\hbar}{2}\Big\{\sum_{l,\,\nu}\big(g^*_{1\nu}\,S^+_l\,R^{a\dagger}_{l\nu}+g_{1\nu}\,S^-_l\,R^a_{l\nu} \;\; +\sum_{m,\,\nu}\big(g^*_{1\nu}\,S^+_m\,R^{b\dagger}_{m\nu}+g_{1\nu}\,S^-_m\,R^b_{m\nu} \;\;\Big\}\\ &-\hbar\,\Big\{\sum_{l,\,\nu}g_{2\nu}\,S^z_l\,R^{a\dagger}_{l\nu}\,R^a_{l\nu}+\sum_{m,\,\nu}g_{2\nu}\,S^z_m\,R^{b\dagger}_{m\nu}\,R^b_{m\nu}\Big\},\end{aligned} \tag{F.1a}$$

$$\begin{aligned}= &-\frac{\hbar}{2}\sum_{k,\,\nu}\big\{\sqrt{2\,S}\,(g^*_{1\nu}\,a_kR^{a\dagger}_{k\nu}+g_{1\nu}\,a^\dagger_kR^a_{k\nu})+\sqrt{2\,S}\,(g^*_{1\nu}\,b^\dagger_kR^{b\dagger}_{k\nu}+g_{1\nu}\,b_kR^b_{k\nu})\big\}+\cdots\\ &-\hbar\sum_{k,\,\nu}g_{2\nu}\Big(S-\frac{2}{N}\sum_{k'}a^\dagger_{k'}a_{k'}\;\;R^{a\dagger}_{k\nu}R^a_{k\nu}-\hbar\sum_{k,\,\nu}g_{2\nu}\Big(\frac{2}{N}\sum_{k'}b^\dagger_{k'}b_{k'}-S\;\;R^{b\dagger}_{k\nu}R^b_{k\nu}+\cdots,\end{aligned} \tag{F.1b}$$

where the first "$\cdots$" of (F.1b) denotes the higher-order parts of the first term of (F.1a) in the spin-wave approximation, and the second "$\cdots$" of (F.1b) denotes the off-diagonal parts in the Fourier transformation of the second term of (F.1a). Assuming that same as the x and y components of the spin, the z component of the spin is coupled only with the phonon operators of the same wave-number as the spin, and renormalizing the free spin-wave Hamiltonian, the free spin-wave energies and the spin-phonon interaction as done in (2.19)−(2.21), the spin-phonon interaction takes the form

$$\begin{aligned}\mathcal{H}_{\mathtt{SR}} = &-\hbar\;\;\overline{S/2}\sum_{k,\,\nu}\big\{g^*_{1\nu}\,(a_kR^{a\dagger}_{k\nu}+b^\dagger_kR^{b\dagger}_{k\nu})+g_{1\nu}\,(a^\dagger_kR^a_{k\nu}+b_kR^b_{k\nu})\big\}\\ &-\hbar\sum_{k,\,\nu}g_{2\nu}\big\{(S-a^\dagger_ka_k)(R^{a\dagger}_{k\nu}R^a_{k\nu}-\langle 1_{\mathtt{R}}|R^{a\dagger}_{k\nu}R^a_{k\nu}|\rho_{\mathtt{R}}\rangle)+(b^\dagger_kb_k-S)(R^{b\dagger}_{k\nu}R^b_{k\nu}-\langle 1_{\mathtt{R}}|R^{b\dagger}_{k\nu}R^b_{k\nu}|\rho_{\mathtt{R}}\rangle)\big\},\end{aligned} \tag{F.2}$$

where the higher-order parts in the spin-wave approximation and the off-diagonal parts and wave-number mixing in $\mathcal{H}_{\mathtt{SR}}$, have been ignored. Substituting (F.2) into (A.19) and by using the basic requirements (A.8) and their tilde conjugates, the collision operator $C^{(2)}$ takes the form [21, 22]

$$\begin{aligned}C^{(2)} = &-S\sum_k\big\{\big\{\phi^{+-}_k(\epsilon^+_k)\{(\alpha_k-\tilde{\alpha}^\dagger_k)\,\alpha^\dagger_k\cosh 2\theta_k-(\beta^\dagger_k-\tilde{\beta}_k)\,\alpha^\dagger_k\sinh 2\theta_k\}\\ &\qquad\qquad+\phi^{+-}_k(-\epsilon^-_k)\{(\beta^\dagger_k-\tilde{\beta}_k)\,\beta_k\cosh 2\theta_k-(\alpha_k-\tilde{\alpha}^\dagger_k)\,\beta_k\sinh 2\theta_k\}\big\}\\ &\qquad-\big\{\phi^{-+}_k(\epsilon^+_k)^*\{(\alpha_k-\tilde{\alpha}^\dagger_k)\,\tilde{\alpha}_k\cosh 2\theta_k-(\beta^\dagger_k-\tilde{\beta}_k)\,\tilde{\alpha}_k\sinh 2\theta_k\}\\ &\qquad\qquad+\phi^{-+}_k(-\epsilon^-_k)^*\{(\beta^\dagger_k-\tilde{\beta}_k)\,\tilde{\beta}^\dagger_k\cosh 2\theta_k-(\alpha_k-\tilde{\alpha}^\dagger_k)\,\tilde{\beta}^\dagger_k\sinh 2\theta_k\}\big\}\\ &\qquad+\big\{\phi^{-+}_k(\epsilon^+_k)\{(\alpha^\dagger_k-\tilde{\alpha}_k)\,\alpha_k\cosh 2\theta_k-(\beta_k-\tilde{\beta}^\dagger_k)\,\alpha_k\sinh 2\theta_k\}\\ &\qquad\qquad+\phi^{-+}_k(-\epsilon^-_k)\{(\beta_k-\tilde{\beta}^\dagger_k)\,\beta^\dagger_k\cosh 2\theta_k-(\alpha^\dagger_k-\tilde{\alpha}_k)\,\beta^\dagger_k\sinh 2\theta_k\}\big\}\\ &\qquad-\big\{\phi^{+-}_k(\epsilon^+_k)^*\{(\alpha^\dagger_k-\tilde{\alpha}_k)\,\tilde{\alpha}^\dagger_k\cosh 2\theta_k-(\beta_k-\tilde{\beta}^\dagger_k)\,\tilde{\alpha}^\dagger_k\sinh 2\theta_k\}\\ &\qquad\qquad+\phi^{+-}_k(-\epsilon^-_k)^*\{(\beta_k-\tilde{\beta}^\dagger_k)\,\tilde{\beta}_k\cosh 2\theta_k-(\alpha^\dagger_k-\tilde{\alpha}_k)\,\tilde{\beta}_k\sinh 2\theta_k\}\big\}\big\}\\ &-\frac{1}{2}\sum_k\big\{\{(\alpha^\dagger_k\alpha_k-\tilde{\alpha}^\dagger_k\tilde{\alpha}_k+\beta^\dagger_k\beta_k-\tilde{\beta}^\dagger_k\tilde{\beta}_k)\cosh 2\theta_k-(\alpha_k\beta_k+\alpha^\dagger_k\beta^\dagger_k-\tilde{\alpha}^\dagger_k\tilde{\beta}^\dagger_k-\tilde{\alpha}_k\tilde{\beta}_k)\sinh 2\theta_k\}\\ &\qquad\times\{(\alpha^\dagger_k\alpha_k-\tilde{\alpha}^\dagger_k\tilde{\alpha}_k+\beta^\dagger_k\beta_k-\tilde{\beta}^\dagger_k\tilde{\beta}_k)\cosh 2\theta_k\,\phi^{zz}_k(0)\\ &\qquad-((\alpha_k\beta_k-\tilde{\alpha}^\dagger_k\tilde{\beta}^\dagger_k)\,\phi^{zz}_k(\epsilon^+_k+\epsilon^-_k)+(\alpha^\dagger_k\beta^\dagger_k-\tilde{\alpha}_k\tilde{\beta}_k)\,\phi^{zz}_k(\epsilon^+_k+\epsilon^-_k)^*)\sinh 2\theta_k\}\\ &\qquad+\{\alpha^\dagger_k\alpha_k-\tilde{\alpha}^\dagger_k\tilde{\alpha}_k-(\beta^\dagger_k\beta_k-\tilde{\beta}^\dagger_k\tilde{\beta}_k)\}\{\alpha^\dagger_k\alpha_k-\tilde{\alpha}^\dagger_k\tilde{\alpha}_k-(\beta^\dagger_k\beta_k-\tilde{\beta}^\dagger_k\tilde{\beta}_k)\}\,\phi^{zz}_k(0)\big\},\end{aligned} \tag{F.3}$$

with the correlation functions $\phi^{+-}_k(\epsilon)$, $\phi^{-+}_k(\epsilon)$ and $\phi^{zz}_k(\epsilon)$ defined by (A.25a) − (A.25c), where the higher-order parts in the spin-wave approximation have been ignored, and the assumption that the phonon correlation function given by

(2.24c) is real, have been used. Then, $n_k^\alpha(t)$ and $n_k^\beta(t)$ defined by (A.32) satisfy the equations of motion

$$\begin{aligned}(d/dt)\, n_k^\alpha(t) &= \langle 1_{\mathrm{S}}|\, U^{-1}(t)\, [\, i\, \hat{\mathcal{H}}_{\mathrm{S}0} - C^{(2)},\, \alpha_k^\dagger\, \alpha_k\,]\, U(t)\, |\rho_0\rangle,\\ &= -\, \{2\, S\, \Phi_k^+(\epsilon_k^+)' \cosh 2\theta_k - \Psi_k' \sinh^2 2\theta_k\}\, n_k^\alpha(t) + \Psi_k' \sinh^2 2\theta_k\, n_k^\beta(t)\\ &\quad + 2\, S\, \Phi_k^+(\epsilon_k^+)' \cosh 2\theta_k\, \bar{n}(\epsilon_k^+) + \Psi_k' \sinh^2 2\theta_k, \qquad \text{(F.4a)}\\ (d/dt)\, n_k^\beta(t) &= \langle 1_{\mathrm{S}}|\, U^{-1}(t)\, [\, i\, \hat{\mathcal{H}}_{\mathrm{S}0} - C^{(2)},\, \beta_k^\dagger\, \beta_k\,]\, U(t)\, |\rho_0\rangle,\\ &= -\, \{2\, S\, \Phi_k^-(\epsilon_k^-)' \cosh 2\theta_k - \Psi_k' \sinh^2 2\theta_k\}\, n_k^\beta(t) + \Psi_k' \sinh^2 2\theta_k\, n_k^\alpha(t)\\ &\quad + 2\, S\, \Phi_k^-(\epsilon_k^-)' \cosh 2\theta_k\, \bar{n}(\epsilon_k^-) + \Psi_k' \sinh^2 2\theta_k, \qquad \text{(F.4b)}\end{aligned}$$

with $\bar{n}(\epsilon_k^\pm)$ defined by (A.41), where $\Phi_k^\pm(\epsilon_k^\pm)'$ and Ψ_k' are the real parts of $\Phi_k^\pm(\epsilon_k^\pm)$ and Ψ_k defined by (A.42) – (A.44), respectively. The quasi-particle operators $\lambda_k(t)$ and $\xi_k(t)$ satisfy the equations of motion

$$\begin{aligned}&(d/dt)\, Z_k^\alpha(t)^{1/2} \langle 1_{\mathrm{S}}|\, \lambda_k(t) = (d/dt)\, \langle 1_{\mathrm{S}}|\, \alpha_k(t) = \langle 1_{\mathrm{S}}|\, U^{-1}(t)\, [\, i\, \hat{\mathcal{H}}_{\mathrm{S}0} - C^{(2)}, \alpha_k\,]\, U(t),\\ &= \{-\, i\, \epsilon_k^+ - S\, \Phi_k^+(\epsilon_k^+) \cosh 2\theta_k - (\Psi_k^0/2)(\cosh^2 2\theta_k + 1) + (\Psi_k/2) \sinh^2 2\theta_k\}\, Z_k^\alpha(t)^{1/2}\, \langle 1_{\mathrm{S}}|\, \lambda_k(t)\\ &\quad -\, \{S\, \Phi_k^-(\epsilon_k^-)^* \sinh 2\theta_k + ((\Psi_k^0 - \Psi_k^*)/2) \sinh 2\theta_k \cosh 2\theta_k\}\, Z_k^\beta(t)^{1/2}\, \langle 1_{\mathrm{S}}|\, \tilde{\xi}_k(t), \qquad \text{(F.5a)}\\ &(d/dt)\, Z_k^\beta(t)^{1/2} \langle 1_{\mathrm{S}}|\, \tilde{\xi}_k(t) = (d/dt)\, \langle 1_{\mathrm{S}}|\, \beta_k^{\dagger\dagger}(t) = \langle 1_{\mathrm{S}}|\, U^{-1}(t)\, [\, i\, \hat{\mathcal{H}}_{\mathrm{S}0} - C^{(2)}, \beta_k^\dagger\,]\, U(t),\\ &= \{i\, \epsilon_k^- - S\, \Phi_k^-(\epsilon_k^-)^* \cosh 2\theta_k - (\Psi_k^0/2)(\cosh^2 2\theta_k + 1) + (\Psi_k^*/2) \sinh^2 2\theta_k\}\, Z_k^\beta(t)^{1/2}\, \langle 1_{\mathrm{S}}|\, \tilde{\xi}_k(t)\\ &\quad -\, \{S\, \Phi_k^+(\epsilon_k^+) \sinh 2\theta_k + (\Psi_k^0 - \Psi_k)/2) \sinh 2\theta_k \cosh 2\theta_k\}\, Z_k^\alpha(t)^{1/2}\, \langle 1_{\mathrm{S}}|\, \lambda_k(t), \qquad \text{(F.5b)}\end{aligned}$$

which correspond to (A.52a) and (A.52b) with $\Gamma_{k\pm}$ and $\Delta_{k\pm}$ given by

$$\Gamma_{k\pm} = S\, \Phi_k^\pm(\epsilon_k^\pm) \cosh 2\theta_k - \Psi_k \sinh^2 2\theta_k/2 + \Psi_k^0 \cdot (\cosh^2 2\theta_k + 1)/2, \qquad \text{(F.6a)}$$

$$\Delta_{k\pm} = S\, \Phi_k^\pm(\epsilon_k^\pm) \sinh 2\theta_k + (\Psi_k^0 - \Psi_k) \sinh 2\theta_k \cosh 2\theta_k/2, \qquad \text{(F.6b)}$$

where Φ_k^0 is defined by (A.54).

The form of interference thermal state $|D_{S_k^-}^{(2)}[\omega]\rangle$ given by (3.8) can be expressed by substituting (F.2) into (3.8) and by using the axioms (A.2), (A.8) and their tilde conjugates, as

$$\begin{aligned}|D_{S_k^-}^{(2)}[\omega]\rangle &= \gamma\, S \quad \overline{S/2}\, (\cosh \theta_k - \sinh \theta_k)\\ &\qquad \times \big\{\{\cosh 2\theta_k\, (\alpha_k^\dagger - \tilde{\alpha}_k) - \sinh 2\theta_k\, (\beta_k - \tilde{\beta}_k^\dagger)\}|\rho_0\rangle\\ &\qquad\quad \times \{(\phi_k^{-+}(\omega) - \phi_k^{+-}(\omega)^*) - (\phi_k^{-+}(\epsilon_k^+) - \phi_k^{+-}(\epsilon_k^+)^*)\}/(\omega - \epsilon_k^+)\\ &\qquad\quad + \{\sinh 2\theta_k\, (\alpha_k^\dagger - \tilde{\alpha}_k) - \cosh 2\theta_k\, (\beta_k - \tilde{\beta}_k^\dagger)\}|\rho_0\rangle\\ &\qquad\quad \times \{(\phi_k^{-+}(\omega) - \phi_k^{+-}(\omega)^*) - (\phi_k^{-+}(-\epsilon_k^-) - \phi_k^{+-}(-\epsilon_k^-)^*)\}/(\omega + \epsilon_k^-)\big\}\\ &\quad + \frac{\gamma\, \sqrt{S}}{2\, \sqrt{2}}\, (\cosh \theta_k - \sinh \theta_k)\\ &\qquad \times \big\{\{(\cosh^2 2\theta_k + 1)\, (\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle\{\phi_k^{zz}(\omega - \epsilon_k^+) - \phi_k^{zz}(0)\}\\ &\qquad\quad + \sinh 2\theta_k \cosh 2\theta_k\, (\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle\{\phi_k^{zz}(\omega + \epsilon_k^-) - \phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-)\}\}/(\omega - \epsilon_k^+)\\ &\qquad\quad + \{\sinh 2\theta_k \cosh 2\theta_k\, (\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle\{\phi_k^{zz}(\omega - \epsilon_k^+) - \phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-)^*\}\\ &\qquad\quad + (\cosh^2 2\theta_k + 1)\, (\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle\{\phi_k^{zz}(\omega + \epsilon_k^-) - \phi_k^{zz}(0)\}\}/(\omega + \epsilon_k^-)\\ &\qquad\quad + (\alpha_k \beta_k + \alpha_k^\dagger \beta_k^\dagger - \tilde{\alpha}_k \tilde{\beta}_k - \tilde{\alpha}_k^\dagger \tilde{\beta}_k^\dagger) \sinh 2\theta_k\\ &\qquad\quad \times \big\{\{\sinh 2\theta_k\, (\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle\{\phi_k^{zz}(\omega + \epsilon_k^-) - \phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-)\}\\ &\qquad\qquad - \cosh 2\theta_k\, (\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle\{\phi_k^{zz}(\omega - \epsilon_k^+) - \phi_k^{zz}(0)\}\}/(\omega - \epsilon_k^+)\\ &\qquad\qquad + \{\cosh 2\theta_k\, (\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle\{\phi_k^{zz}(\omega + \epsilon_k^-) - \phi_k^{zz}(0)\}\\ &\qquad\qquad - \sinh 2\theta_k\, (\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle\{\phi_k^{zz}(\omega - \epsilon_k^+) - \phi_k^{zz}(\epsilon_k^+ + \epsilon_k^-)^*\}\}/(\omega + \epsilon_k^-)\big\}\big\}, \qquad \text{(F.7)}\end{aligned}$$

with the correlation functions $\phi_k^{+-}(\epsilon)$, $\phi_k^{-+}(\epsilon)$ and $\phi_k^{zz}(\epsilon)$ defined by (A.25a) – (A.25c), where the higher-order parts in the spin-wave approximation have been ignored, and the assumption that the phonon correlation function given by (2.24c) is real, have been used. The above expression of the interference thermal state $|D_{S_k^-}^{(2)}[\omega]\rangle$ can be rewritten, by using $\Phi_k^\pm(\epsilon_k^\pm)$, Ψ_k, Ψ_k^0 and $\Psi_k(\epsilon)$ defined by (A.42) – (A.44), (A.54) and (3.12), respectively, as

$$|D_{S_k^-}^{(2)}[\omega]\rangle = \gamma \quad \overline{S/2}\, (\cosh \theta_k - \sinh \theta_k)\{|D_{k1}^{(2)}[\omega]\rangle/(2\, (\omega - \epsilon_k^+)) + |D_{k2}^{(2)}[\omega]\rangle/(2\, (\omega + \epsilon_k^-))\}, \qquad \text{(F.8)}$$

with $|D_{k1}^{(2)}[\omega]\rangle$ and $|D_{k2}^{(2)}[\omega]\rangle$ given by

$$
\begin{aligned}
|D_{k1}^{(2)}[\omega]\rangle = {} & 2\,S\,\{\cosh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k) - \sinh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)\}|\rho_0\rangle\{\Phi_k^+(\omega) - \Phi_k^+(\epsilon_k^+)\} \\
& + \sinh 2\theta_k \cosh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle\{\Psi_k(\omega + \epsilon_k^-) - \Psi_k\} \\
& + (\cosh^2 2\theta_k + 1)\,(\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle\{\Psi_k(\omega - \epsilon_k^+) - \Psi_k^0\} \\
& + (\alpha_k\beta_k + \alpha_k^\dagger\beta_k^\dagger - \tilde{\alpha}_k\tilde{\beta}_k - \tilde{\alpha}_k^\dagger\tilde{\beta}_k^\dagger)\sinh 2\theta_k \\
& \quad \times \big\{ \sinh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle\{\Psi_k(\omega + \epsilon_k^-) - \Psi_k\} \\
& \quad\quad - \cosh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle\{\Psi_k(\omega - \epsilon_k^+) - \Psi_k^0\}\big\}, \qquad \text{(F.9a)} \\
|D_{k2}^{(2)}[\omega]\rangle = {} & 2\,S\,\{\sinh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k) - \cosh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)\}|\rho_0\rangle\{\Phi_k^+(\omega) - \Phi_k^+(-\,\epsilon_k^-)\} \\
& + \sinh 2\theta_k \cosh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle\{\Psi_k(\omega - \epsilon_k^+) - \Psi_k^*\} \\
& + (\cosh^2 2\theta_k + 1)\,(\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle\{\Psi_k(\omega + \epsilon_k^-) - \Psi_k^0\} \\
& + (\alpha_k\beta_k + \alpha_k^\dagger\beta_k^\dagger - \tilde{\alpha}_k\tilde{\beta}_k - \tilde{\alpha}_k^\dagger\tilde{\beta}_k^\dagger)\sinh 2\theta_k \\
& \quad \times \big\{ \cosh 2\theta_k\,(\beta_k - \tilde{\beta}_k^\dagger)|\rho_0\rangle\{\Psi_k(\omega + \epsilon_k^-) - \Psi_k^0\} \\
& \quad\quad - \sinh 2\theta_k\,(\alpha_k^\dagger - \tilde{\alpha}_k)|\rho_0\rangle\{\Psi_k(\omega - \epsilon_k^+) - \Psi_k^*\}\big\}. \qquad \text{(F.9b)}
\end{aligned}
$$

The corresponding interference terms $X_{k1}^\alpha(\omega)$, $X_{k2}^\alpha(\omega)$, $X_{k1}^\beta(\omega)$ and $X_{k2}^\beta(\omega)$, are derived using (C.1), (C.2) and (4.5) – (4.7), and take the following forms:

$$
\begin{aligned}
X_{k1}^\alpha(\omega) &= \langle 1_{\mathbb{S}}|\,\alpha_k\,|D_{k1}^{(2)}[\omega]\rangle/(2\,(\omega - \epsilon_k^+)) = X_{k1}^\alpha(\omega)' + i\,X_{k1}^\alpha(\omega)'', \qquad \text{(F.10a)} \\
&= \big\{2\,S\cosh 2\theta_k\,\{\Phi_k^+(\omega) - \Phi_k^+(\epsilon_k^+)\} + (\cosh^2 2\theta_k + 1)\{\Psi_k(\omega - \epsilon_k^+) - \Psi_k^0\} \\
&\quad - \sinh^2 2\theta_k\,\{\Psi_k(\omega + \epsilon_k^-) - \Psi_k\}\big\}/\{2\,(\omega - \epsilon_k^+)\}, \qquad \text{(F.10b)} \\
&= g_1^2\,S\,\frac{-\,\gamma_{\mathrm{R}k}\,(\omega + \epsilon_k^+ - 2\,\omega_{\mathrm{R}k}) + i\,\{(\gamma_{\mathrm{R}k})^2 - (\omega - \omega_{\mathrm{R}k})(\epsilon_k^+ - \omega_{\mathrm{R}k})\}}{2\,\{(\omega - \omega_{\mathrm{R}k})^2 + (\gamma_{\mathrm{R}k})^2\}\{(\epsilon_k^+ - \omega_{\mathrm{R}k})^2 + (\gamma_{\mathrm{R}k})^2\}}\cosh 2\theta_k \\
&\quad + g_2^2\,\frac{2\,\gamma_{\mathrm{R}k}\,(\omega + \epsilon_k^+ + 2\,\epsilon_k^-) - i\,\{4\,(\gamma_{\mathrm{R}k})^2 - (\omega + \epsilon_k^-)(\epsilon_k^+ + \epsilon_k^-)\}}{2\,\{(\omega + \epsilon_k^-)^2 + 4\,(\gamma_{\mathrm{R}k})^2\}\{(\epsilon_k^+ + \epsilon_k^-)^2 + 4\,(\gamma_{\mathrm{R}k})^2\}}\,\bar{n}(\omega_{\mathrm{R}k})\{\bar{n}(\omega_{\mathrm{R}k}) + 1\}\sinh^2 2\theta_k \\
&\quad + g_2^2\,\frac{-\,(\omega - \epsilon_k^+) + 2\,i\,\gamma_{\mathrm{R}k}}{4\,\gamma_{\mathrm{R}k}\{(\omega - \epsilon_k^+)^2 + 4\,(\gamma_{\mathrm{R}k})^2\}}\,\bar{n}(\omega_{\mathrm{R}k})\{\bar{n}(\omega_{\mathrm{R}k}) + 1\}(\cosh^2 2\theta_k + 1)\,, \qquad \text{(F.10c)}
\end{aligned}
$$

$$
\begin{aligned}
X_{k2}^\alpha(\omega) &= \langle 1_{\mathbb{S}}|\,\alpha_k\,|D_{k2}^{(2)}[\omega]\rangle/(2\,(\omega + \epsilon_k^-)) = X_{k2}^\alpha(\omega)' + i\,X_{k2}^\alpha(\omega)'', \qquad \text{(F.11a)} \\
&= \big\{2\,S\sinh 2\theta_k\,\{\Phi_k^+(\omega) - \Phi_k^+(-\,\epsilon_k^-)\} + \sinh 2\theta_k\cosh 2\theta_k\,\{\Psi_k(\omega - \epsilon_k^+) - \Psi_k^*\} \\
&\quad - \sinh 2\theta_k \cosh 2\theta_k\,\{\Psi_k(\omega + \epsilon_k^-) - \Psi_k^0\}\big\}/\{2\,(\omega + \epsilon_k^-)\}, \qquad \text{(F.11b)} \\
&= g_1^2\,S\,\frac{-\,\gamma_{\mathrm{R}k}\,(\omega - \epsilon_k^- - 2\,\omega_{\mathrm{R}k}) + i\,\{(\gamma_{\mathrm{R}k})^2 + (\omega - \omega_{\mathrm{R}k})(\epsilon_k^- + \omega_{\mathrm{R}k})\}}{2\,\{(\omega - \omega_{\mathrm{R}k})^2 + (\gamma_{\mathrm{R}k})^2\}\{(\epsilon_k^- + \omega_{\mathrm{R}k})^2 + (\gamma_{\mathrm{R}k})^2\}}\sinh 2\theta_k \\
&\quad + g_2^2\,\frac{-\,2\,\gamma_{\mathrm{R}k}\,(\omega - 2\,\epsilon_k^+ - \epsilon_k^-) + i\,\{4\,(\gamma_{\mathrm{R}k})^2 + (\omega - \epsilon_k^+)(\epsilon_k^+ + \epsilon_k^-)\}}{2\,\{(\omega - \epsilon_k^+)^2 + 4\,(\gamma_{\mathrm{R}k})^2\}\{(\epsilon_k^+ + \epsilon_k^-)^2 + 4\,(\gamma_{\mathrm{R}k})^2\}}\,\bar{n}(\omega_{\mathrm{R}k})\{\bar{n}(\omega_{\mathrm{R}k}) + 1\}\sinh 2\theta_k\cosh 2\theta_k \\
&\quad + g_2^2\,\frac{(\omega + \epsilon_k^-) - 2\,i\,\gamma_{\mathrm{R}k}}{4\,\gamma_{\mathrm{R}k}\,\{(\omega + \epsilon_k^-)^2 + 4\,(\gamma_{\mathrm{R}k})^2\}}\,\bar{n}(\omega_{\mathrm{R}k})\{\bar{n}(\omega_{\mathrm{R}k}) + 1\}\sinh 2\theta_k\cosh 2\theta_k\,, \qquad \text{(F.11c)}
\end{aligned}
$$

$$
\begin{aligned}
X_{k1}^\beta(\omega) &= \langle 1_{\mathbb{S}}|\,\beta_k^\dagger\,|D_{k1}^{(2)}[\omega]\rangle/(2\,(\omega - \epsilon_k^+)) = X_{k1}^\beta(\omega)' + i\,X_{k1}^\beta(\omega)'', \qquad \text{(F.12a)} \\
&= \big\{2\,S\sinh 2\theta_k\{\Phi_k^+(\omega) - \Phi_k^+(\epsilon_k^+)\} - \sinh 2\theta_k\cosh 2\theta_k\{\Psi_k(\omega + \epsilon_k^-) - \Psi_k\} \\
&\quad + \sinh 2\theta_k\cosh 2\theta_k\{\Psi_k(\omega - \epsilon_k^+) - \Psi_k^0\}\big\}/\{2\,(\omega - \epsilon_k^+)\}, \qquad \text{(F.12b)} \\
&= g_1^2\,S\,\frac{-\,\gamma_{\mathrm{R}k}\,(\omega + \epsilon_k^+ - 2\,\omega_{\mathrm{R}k}) + i\,\{(\gamma_{\mathrm{R}k})^2 - (\omega - \omega_{\mathrm{R}k})(\epsilon_k^+ - \omega_{\mathrm{R}k})\}}{2\,\{(\omega - \omega_{\mathrm{R}k})^2 + (\gamma_{\mathrm{R}k})^2\}\{(\epsilon_k^+ - \omega_{\mathrm{R}k})^2 + (\gamma_{\mathrm{R}k})^2\}}\sinh 2\theta_k \\
&\quad + g_2^2\,\frac{2\,\gamma_{\mathrm{R}k}\,(\omega + \epsilon_k^+ + 2\,\epsilon_k^-) - i\,\{4\,(\gamma_{\mathrm{R}k})^2 - (\omega + \epsilon_k^-)(\epsilon_k^+ + \epsilon_k^-)\}}{2\,\{(\omega + \epsilon_k^-)^2 + 4\,(\gamma_{\mathrm{R}k})^2\}\{(\epsilon_k^+ + \epsilon_k^-)^2 + 4\,(\gamma_{\mathrm{R}k})^2\}}\,\bar{n}(\omega_{\mathrm{R}k})\{\bar{n}(\omega_{\mathrm{R}k}) + 1\}\sinh 2\theta_k\cosh 2\theta_k \\
&\quad + g_2^2\,\frac{-\,(\omega - \epsilon_k^+) + 2\,i\,\gamma_{\mathrm{R}k}}{4\,\gamma_{\mathrm{R}k}\,\{(\omega - \epsilon_k^+)^2 + 4\,(\gamma_{\mathrm{R}k})^2\}}\,\bar{n}(\omega_{\mathrm{R}k})\{\bar{n}(\omega_{\mathrm{R}k}) + 1\}\sinh 2\theta_k\cosh 2\theta_k\,, \qquad \text{(F.12c)}
\end{aligned}
$$

$$X^{\beta}_{k2}(\omega) = \langle 1_{\mathtt{S}}|\, \beta^{\dagger}_{k}\, |D^{(2)}_{k2}[\omega]\rangle / (2\,(\omega+\epsilon^{-}_{k})) = X^{\beta}_{k2}(\omega)' + i\, X^{\beta}_{k2}(\omega)'', \tag{F.13a}$$

$$= \big\{ 2\,S\, \cosh 2\theta_k\, \{\Phi^{+}_{k}(\omega) - \Phi^{+}_{k}(-\,\epsilon^{-}_{k})\} - (\cosh^2 2\theta_k + 1)\{\Psi_k(\omega+\epsilon^{-}_{k}) - \Psi^{0}_{k}\} + \sinh^2 2\theta_k\, \{\Psi_k(\omega-\epsilon^{+}_{k}) - \Psi^{*}_{k}\}\big\} / \{2\,(\omega+\epsilon^{-}_{k})\}, \tag{F.13b}$$

$$= g_1^2\, S\, \frac{-\,\gamma_{\mathtt{R}k}\,(\omega-\epsilon^{-}_{k}-2\,\omega_{\mathtt{R}k}) + i\,\{(\gamma_{\mathtt{R}k})^2 + (\omega-\omega_{\mathtt{R}k})(\epsilon^{-}_{k}+\omega_{\mathtt{R}k})\}}{2\,\{(\omega-\omega_{\mathtt{R}k})^2+(\gamma_{\mathtt{R}k})^2\}\{(\epsilon^{-}_{k}+\omega_{\mathtt{R}k})^2+(\gamma_{\mathtt{R}k})^2\}}\, \cosh 2\theta_k$$
$$+ g_2^2\, \frac{-\,2\,\gamma_{\mathtt{R}k}\,(\omega-2\,\epsilon^{+}_{k}-\epsilon^{-}_{k}) + i\,\{4\,(\gamma_{\mathtt{R}k})^2 + (\omega-\epsilon^{+}_{k})(\epsilon^{+}_{k}+\epsilon^{-}_{k})\}}{2\,\{(\omega-\epsilon^{+}_{k})^2+4\,(\gamma_{\mathtt{R}k})^2\}\{(\epsilon^{+}_{k}+\epsilon^{-}_{k})^2+4\,(\gamma_{\mathtt{R}k})^2\}}\, \bar{n}(\omega_{\mathtt{R}k})\{\bar{n}(\omega_{\mathtt{R}k})+1\}\sinh^2 2\theta_k$$
$$+ g_2^2\, \frac{(\omega+\epsilon^{-}_{k}) - 2\,i\,\gamma_{\mathtt{R}k}}{4\,\gamma_{\mathtt{R}k}\{(\omega+\epsilon^{-}_{k})^2+4\,(\gamma_{\mathtt{R}k})^2\}}\, \bar{n}(\omega_{\mathtt{R}k})\{\bar{n}(\omega_{\mathtt{R}k})+1\}(\cosh^2 2\theta_k+1)\,. \tag{F.13c}$$

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