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Mathematics Equations to Physics Field Equations --- Sources Moving with Velocity, Acceleration, and Spin

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Abstract

In this article, we show that mathematics can be applied to discover physical laws. First, we show that the following physical laws can be derived from the vector calculus identities: (1) Poynting's Theorem; (2) Gauss's Law for Magnetism; (3) Charge density **C**ontinuity equation**;** (4) Wave equation of potential **A**/electric field **E**/magnetic field **B**; (5) Faraday's Law of Induction; (6) energy density of the magnetic field; (7) time change of energy density of the electric field; (8) time change of energy density of the magnetic field. The above way to rederive physical laws (1) to (8) show the validation and power of the vector calculus identities in discovering physical laws. Then, applying the vector calculus identities, we derive NEW physical field equations for electric charges moving with either uniform velocity **v**, or constant acceleration **a,** or continuous spin **S.** The new physical laws need to be tested experimentally. We suggest that AI can follow the above approach, Mathematical equation → physics theory, to discover new physics laws.

Keyword: vector calculus identities, field equation, electric field, magnetic field, spin

1. Introduction

AI can discover new laws of physics, in the way that AI can identify patterns and rules from experimental data, and to derive physics law from those data:

$$\text{Experiment} \rightarrow \text{physics law.}$$

This approach is basically the approach the physicists historically discovered the physical theory, such as Maxwell theory.

In this article, we show a different approaching: starting with mathematical formulars/identities, then substituting the parameters of physical fields into the mathematical formulars/identities to obtain the physical formulars, the so-obtained formular need to be tested by experiments:

$$\text{Mathematical equation} \rightarrow \text{physics theory} \rightarrow \text{Experimental confirmation}$$

This maybe a new approach for AI to discover new physical laws.

In this article, we start with the vector calculus identities which are a branch of mathematics concerned with differentiation of vector fields (gradient, divergence, curl, and combination of above) [1]. We show that vector calculus identities can be used to analyze how physical quantities, such as electric fields and magnetic fields, change in magnitude and direction across space.

We substitute the parameters of physical fields into vector calculus identities to derive the new physical field equations to describe physical fields, such as electric field and magnetic field. Some of new physical field equations can be confirmed by existing experiments. Some of new physical field equations need to be confirmed experimentally.

The purpose of this article is to derive new physical laws due to the source moving with different motion status. The magnetic field $\mathbf{B}$ is created by current density $\mathbf{j} = \rho\mathbf{v}$, where the velocity $\mathbf{v}$ in vacuum is treated as a constant. By analogy, we list three moving statuses of electrical charges.

Moving statuses of source	Current Density
Stationary	$\mathbf{j} = 0$
Uniform velocity: $\mathbf{v}$	$\mathbf{j} \propto \mathbf{v}$
Constant acceleration: $\mathbf{a}$	$\mathbf{j_a} \propto \mathbf{a}$
Constant spin: $\mathbf{S}$	$\mathbf{j_S} \propto \mathbf{S}$

Note: For the purpose of deriving field equation, we treat the Spin as one of motion status.

Since the velocity of electric charge creates a magnetic field, we propose the following:

(1). The acceleration **a** of electric charge creates a magnetic-type field;

(2). The Spin **S** of electric charge creates a magnetic-type field;

In Section 2, we list the vector calculus identities used in the article.

In Section 3, we show how the vector calculus identities creates the Poynting's Theorem and new physical laws.

In Section 4, we apply vector calculus identities to derive new extended electric field equations and magnetic field equations due to the ***velocity***.

In Section 5, we apply vector calculus identities to derive new extended electric field equations and magnetic field equations due to the ***acceleration***.

In Section 6, we apply vector calculus identities to derive new extended electric field equations and magnetic field equations due to the ***Spin***.

2. Vector Calculus Identities

In Section 2, we list 11 vector calculus identities. Then in Section 3 to Section 6, we utilized them to derive existing and new field equations.

(1). Divergence of cross product of two vector fields: $\mathbf{Y} \times \mathbf{Z}$

$$\nabla \cdot (\mathbf{Y} \times \mathbf{Z}) = (\nabla \times \mathbf{Y}) \cdot \mathbf{Z} - \mathbf{Y} \cdot (\nabla \times \mathbf{Z}) \tag{1}$$

(2) Curl of cross product of two vector fields: $\mathbf{Y} \times \mathbf{Z}$

$$\nabla \times (\mathbf{Y} \times \mathbf{Z}) = \mathbf{Y}(\nabla \cdot \mathbf{Z}) - \mathbf{Z}(\nabla \cdot \mathbf{Y}) + (\mathbf{Z} \cdot \nabla)\mathbf{Y} - (\mathbf{Y} \cdot \nabla)\mathbf{Z} \tag{2}$$

(3) Divergence of Curl of vector field: $\nabla \times \mathbf{Y}$

$$\nabla \cdot (\nabla \times \mathbf{Y}) = 0 \tag{3}$$

(4). Curl of curl of vector field: $\mathbf{Y}$

$$\nabla \times (\nabla \times \mathbf{Y}) = \nabla(\nabla \cdot \mathbf{Y}) - \nabla^2 \mathbf{Y} \tag{4}$$

(5) Divergence of Gradient of scalar field: φ

$$\nabla \cdot (\nabla\varphi) = \nabla^2\varphi \tag{5}$$

(6) Curl of Gradient of scalar field: φ

$$\nabla \times (\nabla\varphi) = 0 \tag{6}$$

(7). Divergence of product of scalar field and vector field: $\varphi\mathbf{Y}$

$$\nabla \cdot (\varphi\mathbf{Y}) = \varphi(\nabla \cdot \mathbf{Y}) + \mathbf{Y} \cdot (\nabla\varphi) \tag{7}$$

(8). Curl of product of scalar field and vector field: $\varphi\mathbf{Y}$

$$\nabla \times (\varphi\mathbf{Y}) = \varphi(\nabla \times \mathbf{Y}) + (\nabla\varphi) \times \mathbf{Y} \tag{8}$$

(9) Gradient of dot product of two vector fields: $(\mathbf{Y} \cdot \mathbf{Z})$**.**

$$\nabla(\mathbf{Y} \cdot \mathbf{Z}) = (\mathbf{Y} \cdot \nabla)\mathbf{Z} + (\mathbf{Z} \cdot \nabla)\mathbf{Y} + \mathbf{Y} \times (\nabla \times \mathbf{Z}) + \mathbf{Z} \times (\nabla \times \mathbf{Y}) \tag{9}$$

(10) Cross Product: $(\mathbf{Y} \times \boldsymbol{\nabla}) \cdot \mathbf{Z}$

$$(\mathbf{Y} \times \boldsymbol{\nabla}) \cdot \mathbf{Z} = \mathbf{Y} \cdot (\nabla \times \mathbf{Z}) \tag{10}$$

(11) Cross Product: $(\mathbf{Y} \times \boldsymbol{\nabla}) \times \mathbf{Z}$

$$(\mathbf{Y} \times \boldsymbol{\nabla}) \times \mathbf{Z} = \mathbf{Y} \times (\nabla \times \mathbf{Z}) + (\mathbf{Y} \cdot \nabla)\mathbf{Z} - \mathbf{Y}(\nabla \cdot \mathbf{Z}) \tag{11}$$

3. Electromagnetism: Electric field and magnetic field

In Section 3, let **Y** and **Z** represent either electric field **E**, or magnetic field **B**, or vector potential **A**, respectively, and considering φ in Equations (5) to Equation (8) as scalar potential. Then substituting **E**, **B**, **A** into vector calculus identities, we obtain the field equations, respectively.

In Section 3, utilizing vector calculus identities of Section 2, we derive the field equations, including:

(1). Poynting's Theorem,

(2). Gauss's Law for Magnetism,

(3). Charge density **C**ontinuity equation**,**

(4). Wave equation of potential A/electric field E/magnetic field B, (5). Faraday's Law of Induction,

(6). energy density of the magnetic field,

(7). time change of energy density of the electric field,

(8). time change of energy density of the magnetic field.

And New field equations.

(3.1). Divergence of cross product of vector fields: E×B, A× B, A×E, and scalar potential φ

(3.1.1) Poynting's Theorem

Substituting **Y** = **E** and **Z** = **B,** Equation (1) gives Poynting's Theorem:

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = (\nabla \times \mathbf{E}) \cdot \mathbf{B} - \mathbf{E} \cdot (\nabla \times \mathbf{B}) =- \frac{1}{2} \partial(\mathbf{E}^2 + \mathbf{B}^2)/\partial t - \mathbf{E} \cdot \mathbf{j} \qquad (3.1)$$

Where $\nabla \times \mathbf{B} = \mathbf{j} + (\partial \mathbf{E}/\partial t)$ and $\nabla \times \mathbf{E} =- (\partial \mathbf{E}/\partial t)$ are used. $\mathbf{E} \times \mathbf{B}$ is the Poynting vector representing the energy flux (power per unit area) of the electromagnetic field. The term, $\partial(\mathbf{E}^2 + \mathbf{B}^2)/\partial t$, is the rate of change of the stored electromagnetic energy density. **E·J** represents the rate at which the electromagnetic field does work on the charge carriers

(3.1.2) New field equation

For completeness, let consider vector potential **A** as well. We obtain new field equations.

Substituting **(Y** = **A** and **Z** = **B)** and **(Y** = **A** and **Z** = **E**), respectively, Equation (1) gives NEW field equation:

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot \mathbf{B} - \mathbf{A} \cdot \mathbf{j} - \mathbf{A} \cdot (\partial \mathbf{E}/\partial t). \qquad (3.2)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{E}) = (\nabla \times \mathbf{A}) \cdot \mathbf{E} - \mathbf{A} \cdot (\nabla \times \mathbf{E}) = \mathbf{E} \cdot \mathbf{B} + \mathbf{A} \cdot (\partial \mathbf{B}/\partial t).. \qquad (3.3)$$

Equation (3.2) represent the energy of magnetic field.

(3.2). Curl of cross product of two vector fields: E×B, A× B, A×E

$$\nabla \times (\mathbf{Y} \times \mathbf{Z}) = \mathbf{Y}(\nabla \cdot \mathbf{Z}) - \mathbf{Z}(\nabla \cdot \mathbf{Y}) + (\mathbf{Z} \cdot \nabla)\mathbf{Y} - (\mathbf{Y} \cdot \nabla)\mathbf{Z} \qquad (2)$$

Substituting **(Y** = **E** and **Z** = **B), (Y** = **A** and **Z** = **B),** and **(Y** = **A** and **Z** = **E),** respectively, Equation (2) gives

NEW field equations:

$$\nabla \times (\mathbf{E} \times \mathbf{B}) = \mathbf{E}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{E}) + (\mathbf{B} \cdot \nabla)\mathbf{E} - (\mathbf{E} \cdot \nabla)\mathbf{B}$$

$$= - q\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{E} - (\mathbf{E} \cdot \nabla)\mathbf{B} \tag{3.4}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

$$= - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{3.5}$$

$$\nabla \times (\mathbf{A} \times \mathbf{E}) = \mathbf{A}(\nabla \cdot \mathbf{E}) - \mathbf{E}(\nabla \cdot \mathbf{A}) + (\mathbf{E} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{E}$$

$$= q\mathbf{A} - \mathbf{E}(\nabla \cdot \mathbf{A}) + (\mathbf{E} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{E} \quad . \tag{3.6}$$

(3.3) Divergence of Curl of vector field: $\nabla \times \mathbf{Y}$

$$\nabla \cdot (\nabla \times \mathbf{Y}) = 0 \tag{3}$$

(3.3.1) Gauss's Law for Magnetism (1)

Substituting **Y = A** and the definition of the magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$, Equation (3) gives Gauss's Law for Magnetism:

$$\nabla \cdot (\nabla \times \mathbf{A}) = \nabla \cdot \mathbf{B} = 0 \tag{3.7}$$

Equation (3.7) mathematically indicates no magnetic monopoles.

(3.3.2) Gauss's Law for Magnetism (2)

Substituting **Y = E,** Equation (3) gives:

$$\nabla \cdot (\nabla \times \mathbf{E}) = 0 \tag{3.8}$$

Substituting the Faraday's Law of Induction into Equation (3.8), we obtain:

$$\nabla \cdot (\nabla \times \mathbf{E}) = - \nabla \cdot (\partial \mathbf{B}/\partial t) = - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0. \tag{3.9a}$$

Equation (3.9a) has two solutions:

(1). The magnetic field **B** is a constant field:

$$\partial \mathbf{B}/\partial t = 0. \tag{3.9b}$$

(2). Gauss's Law for Magnetism

$$\nabla \cdot \mathbf{B}(\mathrm{t}) = 0. \tag{3.9c}$$

Equation (3.9a) is valid for both situations:

(1) constant magnetic field, Equation (3.9b); and

(2) time-varying magnetic field Equation (3.9c).

(3.3.3) Charge density Continuity equation

Substituting **Y = B,** Equation (3) gives:

$$\nabla \cdot \left(\nabla \times \mathbf{B}\right) = 0 \tag{3.10}$$

Substituting the Ampere-Maxwell equation into Equation (3.10), we obtain Charge density Continuity equation,

$$\nabla \cdot \left(\nabla \times \mathbf{B}\right) = \nabla \cdot \mathbf{j} + \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = \nabla \cdot \mathbf{j} + \frac{\partial}{\partial t}\rho = 0 \tag{3.11}$$

(3.4) Curl of curl of vector field: wave equation

$$\nabla \times \left(\nabla \times \mathbf{Y}\right) = \nabla\left(\nabla \cdot \mathbf{Y}\right) - \nabla^2\mathbf{Y} \tag{4}$$

(3.4.1) Wave equation of vector potential A

Substituting **Y = A,** Equation (4) gives:

$$\nabla \times \left(\nabla \times \mathbf{A}\right) = \nabla \times \mathbf{B} = \mathbf{j} + \frac{\partial}{\partial t}\mathbf{E} \tag{3.12a}$$

and

$$\nabla\left(\nabla \cdot \mathbf{A}\right) - \nabla^2\mathbf{A} = -\frac{\partial}{\partial t}\left(\nabla\varphi\right) - \nabla^2\mathbf{A} = \frac{\partial}{\partial t}\mathbf{E} + \frac{\partial^2}{\partial t^2}\mathbf{A} - \nabla^2\mathbf{A} \tag{3.12b}$$

Combining Equation (3.12a) and Equation (3.12b), we obtain wave equation of vector potential **A:**

$$\frac{\partial^2}{\partial t^2}\mathbf{A} - \nabla^2\mathbf{A} = \mathbf{j} \tag{3.12c}$$

Where the definition of **E** field is applied: $\mathbf{E} = -\nabla\varphi - \frac{\partial}{\partial t}\mathbf{A}$**.**

(3.4.2) Wave equation of electric field E

Substituting **Y = E,** Equation (4) gives

$$\nabla\times(\nabla\times\mathbf{E})=-\nabla\times\frac{\partial}{\partial t}\mathbf{B}=-\frac{\partial}{\partial t}\mathbf{j}-\frac{\partial^2}{\partial t^2}\mathbf{E} \tag{3.13a}$$

And

$$\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}=\nabla\rho-\nabla^2\mathbf{E} \tag{3.13b}$$

Combining Equation (3.13a) and Equation (3.13b), we obtain wave equation of electric field **E**,

$$\frac{\partial^2}{\partial t^2}\mathbf{E}-\nabla^2\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{j}-\nabla\rho \tag{3.13c}$$

(3.4.3) Wave equation of magnetic field B

Substituting **Y = B,** Equation (4) gives

$$\nabla\times(\nabla\times\mathbf{B})=\nabla\times\mathbf{j}+\nabla\times\frac{\partial}{\partial t}\mathbf{E}=\nabla\times\mathbf{j}-\frac{\partial^2}{\partial t^2}\mathbf{B} \tag{3.14a}$$

And

$$\nabla(\nabla\cdot\mathbf{B})-\nabla^2\mathbf{B}=-\nabla^2\mathbf{B} \tag{3.14b}$$

Combining Equation (3.14a) and Equation (3.14b), we obtain wave equation of **B**,

$$\frac{\partial^2}{\partial t^2}\mathbf{B}-\nabla^2\mathbf{B}=\nabla\times\mathbf{j} \tag{3.14c}$$

(3.5) Divergence of Gradient of scalar field: wave equation of scalar potential φ

$$\nabla\cdot(\nabla\varphi)=\nabla^2\varphi \tag{5}$$

Substituting the definition of electric field, $\mathbf{E}=-\nabla\varphi-\frac{\partial}{\partial t}\mathbf{A}$ into Equation (5). We obtain wave equation of

potential φ:

$$\nabla\cdot(\nabla\varphi)=\nabla\cdot\left(-\mathbf{E}-\frac{\partial}{\partial t}\mathbf{A}\right)=-\mathrm{q}+\frac{\partial^2}{\partial t^2}\varphi \tag{3.15a}$$

i.e.,

$$\frac{\partial^2}{\partial t^2}\varphi-\nabla^2\varphi=\mathrm{q}. \tag{3.15b}$$

(3.6). Curl of Gradient of scalar field: Faraday's Law of Induction

$$\nabla \times (\nabla\varphi) = 0 \tag{6}$$

Substituting the definition of the electric field, $\mathbf{E} = -\nabla\varphi - \partial\mathbf{A}/\partial t$, into Equation (6).

We obtain Faraday's Law of Induction:

$$\nabla \times (\nabla\varphi) = \nabla \times (-\mathbf{E} - \partial\mathbf{A}/\partial t) = -\nabla \times \mathbf{E} - \partial\mathbf{B}/\partial t = 0.$$
(3.16a)

i.e.,

$$\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t. \tag{3.16b}$$

(3.7). Divergence of product of scalar potential φ with either vector potential A or electric field E or magnetic field B:

$$\nabla \cdot (\varphi\mathbf{Y}) = \varphi(\nabla \cdot \mathbf{Y}) + \mathbf{Y} \cdot (\nabla\varphi) \tag{7}$$

Substituting **Y = A, Y = E,** and **Y = B** into Equation (7), respectively, we obtain NEW field equations:

$$\nabla \cdot (\varphi\mathbf{A}) = \varphi(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla\varphi) \tag{3.17}$$

$$\nabla \cdot (\varphi\mathbf{E}) = \varphi(\nabla \cdot \mathbf{E}) + \mathbf{E} \cdot (\nabla\varphi) = q\varphi + \mathbf{E} \cdot (\nabla\varphi) \tag{3.18}$$

$$\nabla \cdot (\varphi\mathbf{B}) = \varphi(\nabla \cdot \mathbf{B}) + \mathbf{B} \cdot (\nabla\varphi) = \mathbf{B} \cdot (\nabla\varphi) \tag{3.19}$$

(3.8) Curl of product of scalar potential φ and vector field: A, E, B

$$\nabla \times (\varphi\mathbf{Y}) = \varphi(\nabla \times \mathbf{Y}) + (\nabla\varphi) \times \mathbf{Y}. \tag{8}$$

Substituting **Y = A, Y = E,** and **Y = B** into Equation (8), respectively, we obtain NEW field equations**:**

$$\nabla \times (\varphi\mathbf{A}) = \varphi(\nabla \times \mathbf{A}) + (\nabla\varphi) \times \mathbf{A} = \varphi\mathbf{B} + (\nabla\varphi) \times \mathbf{A} \tag{3.20}$$

$$\nabla \times (\varphi\mathbf{E}) = \varphi(\nabla \times \mathbf{E}) + (\nabla\varphi) \times \mathbf{E} = -\varphi\partial\mathbf{B}/\partial t + (\nabla\varphi) \times \mathbf{E} \tag{3.21}$$

$$\nabla \times (\varphi\mathbf{B}) = \varphi(\nabla \times \mathbf{B}) + (\nabla\varphi) \times \mathbf{B} = \mathbf{j}\varphi + \varphi\partial\mathbf{E}/\partial t + (\nabla\varphi) \times \mathbf{B} \tag{3.22}$$

(3.9). Gradient of scalar product of two vector fields: $\nabla(\mathbf{E}\cdot\mathbf{B})$, $\nabla(\mathbf{A}\cdot\mathbf{B})$, and $\nabla(\mathbf{A}\cdot\mathbf{E})$

$$\nabla(\mathbf{Y}\cdot\mathbf{Z}) = (\mathbf{Y}\cdot\nabla)\mathbf{Z} + (\mathbf{Z}\cdot\nabla)\mathbf{Y} + \mathbf{Y}\times(\nabla\times\mathbf{Z}) + \mathbf{Z}\times(\nabla\times\mathbf{Y}). \quad (9)$$

Substituting (**Y = E, Z = B), (Y = A, Z = B)**, **(Y= A, Z = E),** (**Y = Z = B), (Y = Z = A),** ***Y*** **= Z = E)**, into Equation (9), respectively, we obtain NEW field equations:

$$\nabla(\mathbf{E}\cdot\mathbf{B}) = (\mathbf{E}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{E} + \mathbf{E}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{E})$$

$$= (\mathbf{E}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{E} + \mathbf{E}\times\mathbf{j} \quad (3.23)$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$

$$= (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times\mathbf{j} + \mathbf{A}\times\partial\mathbf{E}/\partial\mathrm{t} \quad (3.24)$$

$$\nabla(\mathbf{A}\cdot\mathbf{E}) = (\mathbf{A}\cdot\nabla)\mathbf{E} + (\mathbf{E}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{E}) + \mathbf{E}\times(\nabla\times\mathbf{A})$$

$$= (\mathbf{A}\cdot\nabla)\mathbf{E} + (\mathbf{E}\cdot\nabla)\mathbf{A} - \mathbf{A}\times\partial\mathbf{B}/\partial\mathrm{t} + \mathbf{E}\times\mathbf{B} \quad (3.25)$$

$$\nabla(\mathbf{A}\cdot\mathbf{A}) = 2(\mathbf{A}\cdot\nabla)\mathbf{A} + \mathbf{2A}\times(\nabla\times\mathbf{A})$$

$$= 2(\mathbf{A}\cdot\nabla)\mathbf{A} + \mathbf{2A}\times\mathbf{B} \quad (3.26)$$

$$\nabla(\mathbf{E}\cdot\mathbf{E}) = (\mathbf{E}\cdot\nabla)\mathbf{E} + (\mathbf{E}\cdot\nabla)\mathbf{E} + \mathbf{E}\times(\nabla\times\mathbf{E}) + \mathbf{E}\times(\nabla\times\mathbf{E})$$

$$= 2(\mathbf{E}\cdot\nabla)\mathbf{E} - 2\mathbf{E}\times(\partial\mathbf{B}/\partial\mathrm{t}) \quad (3.27)$$

$$\nabla(\mathbf{B}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{B})$$

$$= 2(\mathbf{B}\cdot\nabla)\mathbf{B} + \mathbf{2B}\times(\nabla\times\mathbf{B}). \quad (3.28)$$

(3.10). Cross Product: $(\mathbf{Y}\times\boldsymbol{\nabla})\cdot\mathbf{Z}$

Equation (10),

$$(\mathbf{Y}\times\boldsymbol{\nabla})\cdot\mathbf{Z} = \mathbf{Y}\cdot(\nabla\times\mathbf{Z}), \quad (10)$$

gives the power density and the energy density of the electric field

Substituting **(Y = E** and **Z = B), (Y = B** and **Z = E), (Y = A** and **Z = B), (Y = B** and **Z = A), (Y = E** and **Z = A),** and **(Y = A** and **Z = E), respectively,** Equation (10) gives the following equations,

$$(\mathbf{E} \times \boldsymbol{\nabla}) \cdot \mathbf{B} = \mathbf{E} \cdot (\nabla \times \mathbf{B}) = \mathbf{E} \cdot \mathbf{j} + \mathbf{E} \cdot \partial \mathbf{E}/\partial \mathrm{t} \tag{3.29}$$

$$(\mathbf{B} \times \boldsymbol{\nabla}) \cdot \mathbf{E} = \mathbf{B} \cdot (\nabla \times \mathbf{E}) = -\mathbf{B} \cdot \partial \mathbf{B}/\partial \mathrm{t}. \tag{3.30}$$

$$(\mathbf{A} \times \boldsymbol{\nabla}) \cdot \mathbf{B} = \mathbf{A} \cdot (\nabla \times \mathbf{B}) = \mathbf{A} \cdot \mathbf{j} + \mathbf{A} \cdot \partial \mathbf{E}/\partial \mathrm{t} \tag{3.31}$$

$$(\mathbf{B} \times \boldsymbol{\nabla}) \cdot \mathbf{A} = \mathbf{B} \cdot (\nabla \times \mathbf{A}) = \mathbf{B}^2 \tag{3.32}$$

$$(\mathbf{E} \times \boldsymbol{\nabla}) \cdot \mathbf{A} = \mathbf{E} \cdot (\nabla \times \mathbf{A}) = \mathbf{E} \cdot \mathbf{B} \tag{3.33}$$

$$(\mathbf{A} \times \boldsymbol{\nabla}) \cdot \mathbf{E} = \mathbf{A} \cdot (\nabla \times \mathbf{E}) = -\mathbf{A} \cdot \partial \mathbf{B}/\partial \mathrm{t} \tag{3.34}$$

(3.11). Cross Product: $(\mathbf{Y} \times \boldsymbol{\nabla}) \times \mathbf{Z}$

$$(\mathbf{Y} \times \boldsymbol{\nabla}) \times \mathbf{Z} = \mathbf{Y} \times (\nabla \times \mathbf{Z}) + (\mathbf{Y} \cdot \nabla)\mathbf{Z} - \mathbf{Y}(\nabla \cdot \mathbf{Z}) \tag{11}$$

Substituting **($\mathbf{Y} = \mathbf{E}$ and $\mathbf{Z} = \mathbf{B}$), ($\mathbf{Y} = \mathbf{B}$ and $\mathbf{Z} = \mathbf{E}$), ($\mathbf{Y} = \mathbf{A}$ and $\mathbf{Z} = \mathbf{B}$), ($\mathbf{Y} = \mathbf{B}$ and $\mathbf{Z} = \mathbf{A}$), ($\mathbf{Y} = \mathbf{A}$ and $\mathbf{Z} = \mathbf{E}$),** and **($\mathbf{Y} = \mathbf{E}$ and $\mathbf{Z} = \mathbf{A}$),** Equation (11) gives new field equations:

$$(\mathbf{E} \times \boldsymbol{\nabla}) \times \mathbf{B} = \mathbf{E} \times (\nabla \times \mathbf{B}) + (\mathbf{E} \cdot \nabla)\mathbf{B} - \mathbf{E}(\nabla \cdot \mathbf{B}) = \mathbf{E} \times \mathbf{j} + \mathbf{E} \times \partial \mathbf{E}/\partial \mathrm{t} + (\mathbf{E} \cdot \nabla)\mathbf{B}$$

$$= \mathbf{E} \times \mathbf{j} + (\mathbf{E} \cdot \nabla)\mathbf{B} \tag{3.35}$$

$$(\mathbf{B} \times \boldsymbol{\nabla}) \times \mathbf{E} = \mathbf{B} \times (\nabla \times \mathbf{E}) + (\mathbf{B} \cdot \nabla)\mathbf{E} - \mathbf{B}(\nabla \cdot \mathbf{E}) = -\mathbf{B} \times \partial \mathbf{B}/\partial \mathrm{t} + (\mathbf{B} \cdot \nabla)\mathbf{E} - \mathrm{q}\mathbf{B}$$

$$= -\mathrm{q}\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{E} \tag{3.36}$$

$$(\mathbf{A} \times \boldsymbol{\nabla}) \times \mathbf{B} = \mathbf{A} \times (\nabla \times \mathbf{B}) + (\mathbf{A} \cdot \nabla)\mathbf{B} - \mathbf{A}(\nabla \cdot \mathbf{B})$$

$$= \mathbf{A} \times \mathbf{j} + \mathbf{A} \times \partial \mathbf{E}/\partial \mathrm{t} + (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{3.37}$$

$$(\mathbf{B} \times \boldsymbol{\nabla}) \times \mathbf{A} = \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A})$$

$$= (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{B}(\partial \varphi/\partial \mathrm{t}) \tag{3.38}$$

$$(\mathbf{A} \times \boldsymbol{\nabla}) \times \mathbf{E} = \mathbf{A} \times (\nabla \times \mathbf{E}) + (\mathbf{A} \cdot \nabla)\mathbf{E} - \mathbf{A}(\nabla \cdot \mathbf{E})$$

$$= -\mathbf{A} \times \partial \mathbf{B}/\partial \mathrm{t} + (\mathbf{A} \cdot \nabla)\mathbf{E} - \mathrm{q}\mathbf{A} \tag{3.39}$$

$$(\mathbf{E} \times \boldsymbol{\nabla}) \times \mathbf{A} = \mathbf{E} \times \mathbf{B} + (\mathbf{E} \cdot \nabla)\mathbf{A} - \mathbf{E}(\nabla \cdot \mathbf{A})$$

$$= \mathbf{E} \times \mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{E}(\partial \varphi/\partial \mathrm{t}) \tag{3.40}$$

In this Section, we have ***mathematically*** derived well-known physics laws:

(a) Poynting's Theorem

(b) Gauss's Law for Magnetism

(c) Charge density continuity equation

(d) Faraday's Law of Induction

(e) Wave equation of vector potential **A**, electric field **E** and magnetic field **B**

(f) Energy density of time change of the electric field

(g) Energy density of time change of the magnetic field

(h) Energy density of the magnetic field

We also derived ***mathematically*** several new physical laws.

Summary of Section 3: Relation between Vector potential **A**, Electric field **E** and magnetic field **B**

Vector Calculus Identities	**A, E** and **B**
$\nabla\cdot(\mathbf{Y}\times\mathbf{Z})=(\nabla\times\mathbf{Y})\cdot\mathbf{Z}-\mathbf{Y}\cdot(\nabla\times\mathbf{Z})$	$\nabla\cdot(\mathbf{E}\times\mathbf{B})=-(1/2)(\partial(\mathbf{E}^2+\mathbf{B}^2)/\partial t-\mathbf{E}\cdot\mathbf{j}$ $\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\mathbf{B}-\mathbf{A}\cdot\mathbf{j}-\mathbf{A}\cdot(\partial\mathbf{E}/\partial t)$. $\nabla\cdot(\mathbf{A}\times\mathbf{E})=\mathbf{E}\cdot\mathbf{B}+\mathbf{A}\cdot(\partial\mathbf{B}/\partial t)$
$\nabla\times(\mathbf{Y}\times\mathbf{Z})=\mathbf{Y}(\nabla\cdot\mathbf{Z})-\mathbf{Z}(\nabla\cdot\mathbf{Y})$ $+(\mathbf{Z}\cdot\nabla)\mathbf{Y}-(\mathbf{Y}\cdot\nabla)\mathbf{Z}$	$\nabla\times(\mathbf{E}\times\mathbf{B})=-\mathrm{q}\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{E}-(\mathbf{E}\cdot\nabla)\mathbf{B}$ $\nabla\times(\mathbf{A}\times\mathbf{B})=-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$ $\nabla\times(\mathbf{A}\times\mathbf{E})=\mathrm{q}\mathbf{A}-\mathbf{E}(\nabla\cdot\mathbf{A})+(\mathbf{E}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{E}$
$\nabla\cdot(\nabla\times\mathbf{Y})=0$	$\nabla\cdot(\nabla\times\mathbf{A})=\nabla\cdot\mathbf{B}=0$ $\nabla\cdot(\nabla\times\mathbf{E})=-\partial(\nabla\cdot\mathbf{B})/\partial\mathrm{t}=0$ $\nabla\cdot(\nabla\times\mathbf{B})=\nabla\cdot\mathbf{j}+\partial\rho/\partial\mathrm{t}=0$
$\nabla\times(\nabla\times\mathbf{Y})=\nabla(\nabla\cdot\mathbf{Y})-\nabla^2\mathbf{Y}$	$\frac{\partial^2}{\partial t^2}\mathbf{A}-\nabla^2\mathbf{A}=\mathbf{j}$ $\frac{\partial^2}{\partial t^2}\mathbf{E}-\nabla^2\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{j}-\nabla\rho$ $\frac{\partial^2}{\partial t^2}\mathbf{B}-\nabla^2\mathbf{B}=\nabla\times\mathbf{j}$
$\nabla\cdot(\nabla\varphi)=\nabla^2\varphi$	$\frac{\partial^2}{\partial t^2}\varphi-\nabla^2\varphi=\mathrm{q}$.
$\nabla\times(\nabla\varphi)=0$	$\nabla\times(\nabla\varphi)=\nabla\times(-\mathbf{E}-\partial\mathbf{A}/\partial\mathrm{t})=-\nabla\times\mathbf{E}-\partial\mathbf{B}/\partial\mathrm{t}=0$
$\nabla\cdot(\varphi\mathbf{Y})=\varphi(\nabla\cdot\mathbf{Y})+\mathbf{Y}\cdot(\nabla\varphi)$	$\nabla\cdot(\varphi\mathbf{A})=\varphi(\nabla\cdot\mathbf{A})+\mathbf{A}\cdot(\nabla\varphi)$ $\nabla\cdot(\varphi\mathbf{E})=\mathrm{q}\varphi+\mathbf{E}\cdot(\nabla\varphi)$

	$\nabla\cdot(\varphi\mathbf{B}) = 0 + \mathbf{B}\cdot(\nabla\varphi)$
$\nabla\times(\varphi\mathbf{Y}) = \varphi(\nabla\times\mathbf{Y}) + (\nabla\varphi)\times\mathbf{Y}$	$\nabla\times(\varphi\mathbf{A}) = \varphi(\nabla\times\mathbf{A}) + (\nabla\varphi)\times\mathbf{A}$ $\nabla\times(\varphi\mathbf{E}) =-\varphi\partial\mathbf{B}/\partial t + (\nabla\varphi)\times\mathbf{E}$ $\nabla\times(\varphi\mathbf{B}) = \mathbf{j}\varphi + \varphi\partial\mathbf{E}/\partial t + (\nabla\varphi)\times\mathbf{B}$
$\nabla(\mathbf{Y}\cdot\mathbf{Z}) = (\mathbf{Y}\cdot\nabla)\mathbf{Z} + (\mathbf{Z}\cdot\nabla)\mathbf{Y}$ $+\mathbf{Y}\times(\nabla\times\mathbf{Z}) + \mathbf{Z}\times(\nabla\times\mathbf{Y})$	$\nabla(\mathbf{E}\cdot\mathbf{B}) = (\mathbf{E}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{E} + \mathbf{E}\times\mathbf{j}$ $\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times\mathbf{j} + \mathbf{A}\times\partial\mathbf{E}/\partial t$ $\nabla(\mathbf{A}\cdot\mathbf{E}) = (\mathbf{A}\cdot\nabla)\mathbf{E} + (\mathbf{E}\cdot\nabla)\mathbf{A} - \mathbf{A}\times\partial\mathbf{B}/\partial t + \mathbf{E}\times\mathbf{B}$ $\nabla(\mathbf{A}\cdot\mathbf{A}) = 2(\mathbf{A}\cdot\nabla)\mathbf{A} + \mathbf{2A}\times\mathbf{B}$ $\nabla(\mathbf{E}\cdot\mathbf{E}) = 2(\mathbf{E}\cdot\nabla)\mathbf{E} - 2\mathbf{E}\times(\partial\mathbf{B}/\partial t)$ $\nabla(\mathbf{B}\cdot\mathbf{B}) = 2(\mathbf{B}\cdot\nabla)\mathbf{B} + \mathbf{2B}\times(\nabla\times\mathbf{B})$.
$(\mathbf{Y}\times\boldsymbol{\nabla})\cdot\mathbf{Z} = \mathbf{Y}\cdot(\nabla\times\mathbf{Z})$	$(\mathbf{E}\times\boldsymbol{\nabla})\cdot\mathbf{B} = \mathbf{E}\cdot(\nabla\times\mathbf{B}) = \mathbf{E}\cdot\mathbf{j} + \mathbf{E}\cdot\partial\mathbf{E}/\partial t$ $(\mathbf{B}\times\boldsymbol{\nabla})\cdot\mathbf{E} = \mathbf{B}\cdot(\nabla\times\mathbf{E}) =-\mathbf{B}\cdot\partial\mathbf{B}/\partial t$ $(\mathbf{A}\times\boldsymbol{\nabla})\cdot\mathbf{B} = \mathbf{A}\cdot\mathbf{j} + \mathbf{A}\cdot\partial\mathbf{E}/\partial t$ $(\mathbf{B}\times\boldsymbol{\nabla})\cdot\mathbf{A} = \mathbf{B}^2$ $(\mathbf{A}\times\boldsymbol{\nabla})\cdot\mathbf{E} =-\mathbf{A}\cdot\partial\mathbf{B}/\partial t$ $(\mathbf{E}\times\boldsymbol{\nabla})\cdot\mathbf{A} = \mathbf{E}\cdot\mathbf{B}$
$(\mathbf{Y}\times\boldsymbol{\nabla})\times\mathbf{Z} = \mathbf{Y}\times(\nabla\times\mathbf{Z}) + (\mathbf{Y}\cdot\nabla)\mathbf{Z}$ $-\mathbf{Y}(\nabla\cdot\mathbf{Z})$	$(\mathbf{E}\times\boldsymbol{\nabla})\times\mathbf{B} = \mathbf{E}\times\mathbf{j} + (\mathbf{E}\cdot\nabla)\mathbf{B}$ $(\mathbf{B}\times\boldsymbol{\nabla})\times\mathbf{E} =-q\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{E}$ $(\mathbf{B}\times\boldsymbol{\nabla})\times\mathbf{A} = (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{B}(\partial\varphi/\partial t)$ $(\mathbf{A}\times\boldsymbol{\nabla})\times\mathbf{B} = \mathbf{A}\times\mathbf{j} + \mathbf{A}\times\partial\mathbf{E}/\partial t + (\mathbf{A}\cdot\nabla)\mathbf{B}$ $(\mathbf{A}\times\boldsymbol{\nabla})\times\mathbf{E} =-\mathbf{A}\times\partial\mathbf{B}/\partial t + (\mathbf{A}\cdot\nabla)\mathbf{E} - q\mathbf{A}$ $(\mathbf{E}\times\boldsymbol{\nabla})\times\mathbf{A} = \mathbf{E}\times\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{E}(\partial\varphi/\partial t)$

4. Electric field and magnetic field created by e-charge moving with velocity v

(4.1). Divergence of cross product of velocity v and vector fields, A, E, B

$$\nabla\cdot(\mathbf{Y}\times\mathbf{Z}) = (\nabla\times\mathbf{Y})\cdot\mathbf{Z} - \mathbf{Y}\cdot(\nabla\times\mathbf{Z}) \tag{1}$$

Substituting **(Y** = **v** and **Z** = **A,** vector potential), **(Y** = **v** and **Z** = **E,** electric field**),** and (**Y** = **v** and **Z** = **B,** magnetic field**),** respectively, Equation (1) gives New field equations:

$$\nabla\cdot(\mathbf{v}\times\mathbf{A}) = (\nabla\times\mathbf{v})\cdot\mathbf{A} - \mathbf{v}\cdot(\nabla\times\mathbf{A}) =-\mathbf{v}\cdot\mathbf{B}. \tag{4.1}$$

$$\nabla\cdot(\mathbf{v}\times\mathbf{E}) = (\nabla\times\mathbf{v})\cdot\mathbf{E} - \mathbf{v}\cdot(\nabla\times\mathbf{E}) = \mathbf{v}\cdot\partial(\mathbf{B})/\partial t. \tag{4.2}$$

$$\nabla\cdot(\mathbf{v}\times\mathbf{B}) = (\nabla\times\mathbf{v})\cdot\mathbf{B} - \mathbf{v}\cdot(\nabla\times\mathbf{B}) =-\mathbf{v}\cdot\mathbf{j} - \mathbf{v}\cdot\partial(\mathbf{E})/\partial t \tag{4.3}$$

Where v is particles' average drift velocity.

(4.2). Curl of cross product of velocity v and vector fields, A, E, B

$$\nabla\times(\mathbf{Y}\times\mathbf{Z})=\mathbf{Y}(\nabla\cdot\mathbf{Z})-\mathbf{Z}(\nabla\cdot\mathbf{Y})+(\mathbf{Z}\cdot\nabla)\mathbf{Y}-(\mathbf{Y}\cdot\nabla)\mathbf{Z} \tag{2}$$

Substituting **(Y** = **v, Z** = **A,** vector potential), **(Y** = **v**, **Z** = **E,** electric field**),** and (**Y** = **v**, **Z** = **B,** magnetic field**),** respectively, Equation (2) gives NEW field equations:

$$\nabla\times(\mathbf{v}\times\mathbf{A})=\mathbf{v}(\nabla\cdot\mathbf{A})-\mathbf{A}(\nabla\cdot\mathbf{v})+(\mathbf{A}\cdot\nabla)\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{A}=\mathbf{v}(\nabla\cdot\mathbf{A})-\partial(\mathbf{A})/\partial\mathrm{t}. \tag{4.4}$$

$$\nabla\times(\mathbf{v}\times\mathbf{E})=\mathbf{v}(\nabla\cdot\mathbf{E})-\mathbf{E}(\nabla\cdot\mathbf{v})+(\mathbf{E}\cdot\nabla)\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{E}=\mathrm{q}\mathbf{v}-\partial(\mathbf{E})/\partial\mathrm{t} \tag{4.5}$$

$$\nabla\times(\mathbf{v}\times\mathbf{B})=\mathbf{v}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{v})+(\mathbf{B}\cdot\nabla)\mathbf{v}-(\mathbf{v}\cdot\nabla)\mathbf{B}=-\partial(\mathbf{B})/\partial\mathrm{t}. \tag{4.6}$$

(4.3). Divergence of product of scalar potential φ and velocity v

$$\nabla\cdot(\varphi\mathbf{Y})=\varphi(\nabla\cdot\mathbf{Y})+\mathbf{Y}\cdot(\nabla\varphi) \tag{7}$$

Substituting **Y = v** into Equation (7), we obtain NEW field equation

$$\nabla\cdot(\varphi\mathbf{v})=\varphi(\nabla\cdot\mathbf{v})+\mathbf{v}\cdot(\nabla\varphi)=\mathbf{v}\cdot(\nabla\varphi) \tag{4.7}$$

(4.4) Curl of product of scalar field φ and velocity v

$$\nabla\times(\varphi\mathbf{Y})=\varphi(\nabla\times\mathbf{Y})+(\nabla\varphi)\times\mathbf{Y}. \tag{8}$$

Substituting **Y = v** into Equation (8), we obtain NEW field equation:

$$\nabla\times(\varphi\mathbf{v})=\varphi(\nabla\times\mathbf{v})+(\nabla\varphi)\times\mathbf{v}=(\nabla\varphi)\times\mathbf{v} \tag{4.8}$$

(4.5). Gradient of scalar product of velocity v and vector fields, A, E, B

$$\nabla(\mathbf{Y}\cdot\mathbf{Z})=(\mathbf{Y}\cdot\nabla)\mathbf{Z}+(\mathbf{Z}\cdot\nabla)\mathbf{Y}+\mathbf{Y}\times(\nabla\times\mathbf{Z})+\mathbf{Z}\times(\nabla\times\mathbf{Y}). \tag{9}$$

Substituting (**Y = v, Z = A**), or (**Y = v, Z = E**), or (**Y = v, Z = B**), into Equation (9), respectively, we obtain NEW field equations:

$$\nabla(\mathbf{v}\cdot\mathbf{A})=(\mathbf{v}\cdot\nabla)\mathbf{A}+(\mathbf{A}\cdot\nabla)\mathbf{v}+\mathbf{v}\times(\nabla\times\mathbf{A})+\mathbf{A}\times(\nabla\times\mathbf{v})$$

$$=(\mathbf{v}\cdot\nabla)\mathbf{A}+\mathbf{v}\times\mathbf{B} \tag{4.9}$$

$$\nabla(\mathbf{v}\cdot\mathbf{E})=(\mathbf{v}\cdot\nabla)\mathbf{E}+(\mathbf{E}\cdot\nabla)\mathbf{v}+\mathbf{v}\times(\nabla\times\mathbf{E})+\mathbf{E}\times(\nabla\times\mathbf{v})$$

$$=(\mathbf{v}\cdot\nabla)\mathbf{E}+\mathbf{v}\times(\nabla\times\mathbf{E}). \tag{4.10}$$

$$\nabla(\mathbf{v}\cdot\mathbf{B})=(\mathbf{v}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{v}+\mathbf{v}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{v})$$

$$= (\mathbf{v}\cdot\nabla)\mathbf{B} + \mathbf{v}\times(\nabla\times\mathbf{B}). \tag{4.11}$$

(4.6). Cross Product: $(\mathbf{Y}\times\boldsymbol{\nabla})\cdot\mathbf{Z}$

$$(\mathbf{Y}\times\boldsymbol{\nabla})\cdot\mathbf{Z} = \mathbf{Y}\cdot(\nabla\times\mathbf{Z}). \tag{10}$$

Substituting **($\mathbf{Y}=\mathbf{v}, \mathbf{Z}=\mathbf{A}$)**, or **($\mathbf{Y}=\mathbf{v}, \mathbf{Z}=\mathbf{E}$),** or **($\mathbf{Y}=\mathbf{v}, \mathbf{Z}=\mathbf{B}$),** respectively, Equation (10) gives new field equations,

$$(\mathbf{v}\times\boldsymbol{\nabla})\cdot\mathbf{A} = \mathbf{v}\cdot(\nabla\times\mathbf{A}) = \mathbf{v}\cdot\mathbf{B}. \tag{4.12}$$

$$(\mathbf{v}\times\boldsymbol{\nabla})\cdot\mathbf{E} = \mathbf{v}\cdot(\nabla\times\mathbf{E}) =-\mathbf{v}\cdot\partial\mathbf{B}/\partial\mathrm{t}. \tag{4.13}$$

$$(\mathbf{v}\times\boldsymbol{\nabla})\cdot\mathbf{B} = \mathbf{v}\cdot(\nabla\times\mathbf{B}) = \mathbf{v}\cdot\mathbf{j} + \mathbf{v}\cdot\partial\mathbf{E}/\partial\mathrm{t}. \tag{4.14}$$

(4.7). Cross Product: $(\mathbf{Y}\times\boldsymbol{\nabla})\times\mathbf{Z}$

$$(\mathbf{Y}\times\boldsymbol{\nabla})\times\mathbf{Z} = \mathbf{Y}\times(\nabla\times\mathbf{Z}) + (\mathbf{Y}\cdot\nabla)\mathbf{Z} - \mathbf{Y}(\nabla\cdot\mathbf{Z}) \tag{11}$$

Substituting **($\mathbf{Y}=\mathbf{v}, \mathbf{Z}=\mathbf{A}$)**, or **($\mathbf{Y}=\mathbf{v}, \mathbf{Z}=\mathbf{E}$),** or **($\mathbf{Y}=\mathbf{v}, \mathbf{Z}=\mathbf{B}$),** respectively, Equation (11) gives

new field equations:

$$(\mathbf{v}\times\boldsymbol{\nabla})\times\mathbf{A} = \mathbf{v}\times(\nabla\times\mathbf{A}) + (\mathbf{v}\cdot\nabla)\mathbf{A} - \mathbf{v}(\nabla\cdot\mathbf{A}) = \mathbf{v}\times\mathbf{B} + \partial\mathbf{A}/\partial\mathrm{t} - \mathbf{v}(\nabla\cdot\mathbf{A}). \tag{4.15}$$

$$(\mathbf{v}\times\boldsymbol{\nabla})\times\mathbf{E} = \mathbf{v}\times(\nabla\times\mathbf{E}) + (\mathbf{v}\cdot\nabla)\mathbf{E} - \mathbf{v}(\nabla\cdot\mathbf{E}) = \mathbf{v}\times(\nabla\times\mathbf{E}) + \partial\mathbf{E}/\partial\mathrm{t} - \mathrm{q}\mathbf{v}. \tag{4.16}$$

$$(\mathbf{v}\times\boldsymbol{\nabla})\times\mathbf{B} = \mathbf{v}\times(\nabla\times\mathbf{B}) + (\mathbf{v}\cdot\nabla)\mathbf{B} - \mathbf{v}(\nabla\cdot\mathbf{B}) = \mathbf{v}\times(\nabla\times\mathbf{B}) + \partial\mathbf{B}/\partial\mathrm{t}. \tag{4.17}$$

Summary of Section 4: Electric field and magnetic field created by electric charge moving at velocity **v**

Vector Calculus Identities	**v, A, E** and **B**
$\nabla\cdot(\mathbf{Y}\times\mathbf{Z}) = (\nabla\times\mathbf{Y})\cdot\mathbf{Z} - \mathbf{Y}\cdot(\nabla\times\mathbf{Z})$	$\nabla\cdot(\mathbf{v}\times\mathbf{A}) =-\mathbf{v}\cdot\mathbf{B}$ $\nabla\cdot(\mathbf{v}\times\mathbf{E}) = \mathbf{v}\cdot\partial(\mathbf{B})/\partial\mathrm{t}.$ $\nabla\cdot(\mathbf{v}\times\mathbf{B}) =-\mathbf{v}\cdot\mathbf{j} - \mathbf{v}\cdot\partial(\mathbf{E})/\partial\mathrm{t}$
$\nabla\times(\mathbf{Y}\times\mathbf{Z}) = \mathbf{Y}(\nabla\cdot\mathbf{Z}) - \mathbf{Z}(\nabla\cdot\mathbf{Y})$ $+ (\mathbf{Z}\cdot\nabla)\mathbf{Y} - (\mathbf{Y}\cdot\nabla)\mathbf{Z}$	$\nabla\times(\mathbf{v}\times\mathbf{A}) = \mathbf{v}(\nabla\cdot\mathbf{A}) - \partial(\mathbf{A})/\partial\mathrm{t}.$ $\nabla\times(\mathbf{v}\times\mathbf{E}) = \mathrm{q}\mathbf{v} - \partial(\mathbf{E})/\partial\mathrm{t}$ $\nabla\times(\mathbf{v}\times\mathbf{B}) =-\partial(\mathbf{B})/\partial\mathrm{t}.$
$\nabla\cdot(\varphi\mathbf{Y}) = \varphi(\nabla\cdot\mathbf{Y}) + \mathbf{Y}\cdot(\nabla\varphi)$	$\nabla\cdot(\varphi\mathbf{v}) = \mathbf{v}\cdot(\nabla\varphi)$
$\nabla\times(\varphi\mathbf{Y}) = \varphi(\nabla\times\mathbf{Y}) + (\nabla\varphi)\times\mathbf{Y}$	$\nabla\times(\varphi\mathbf{v}) = (\nabla\varphi)\times\mathbf{v}$

$\nabla(\mathbf{Y}\cdot\mathbf{Z}) = (\mathbf{Y}\cdot\nabla)\mathbf{Z} + (\mathbf{Z}\cdot\nabla)\mathbf{Y}$ $+\mathbf{Y}\times(\nabla\times\mathbf{Z}) + \mathbf{Z}\times(\nabla\times\mathbf{Y})$	$\nabla(\mathbf{v}\cdot\mathbf{A}) = (\mathbf{v}\cdot\nabla)\mathbf{A} + \mathbf{v}\times\mathbf{B}$ $\nabla(\mathbf{v}\cdot\mathbf{E}) = (\mathbf{v}\cdot\nabla)\mathbf{E} + \mathbf{v}\times(\nabla\times\mathbf{E})$ $\nabla(\mathbf{v}\cdot\mathbf{B}) = (\mathbf{v}\cdot\nabla)\mathbf{B} + \mathbf{v}\times(\nabla\times\mathbf{B})$
$(\mathbf{Y}\times\boldsymbol{\nabla})\cdot\mathbf{Z} = \mathbf{Y}\cdot(\nabla\times\mathbf{Z})$	$(\mathbf{v}\times\boldsymbol{\nabla})\cdot\mathbf{A} = \mathbf{v}\cdot\mathbf{B}$. $(\mathbf{v}\times\boldsymbol{\nabla})\cdot\mathbf{E} =-\mathbf{v}\cdot\partial\mathbf{B}/\partial\mathrm{t}$. $(\mathbf{v}\times\boldsymbol{\nabla})\cdot\mathbf{B} = \mathbf{v}\cdot\mathbf{j} + \mathbf{v}\cdot\partial\mathbf{E}/\partial\mathrm{t}$.
$(\mathbf{Y}\times\boldsymbol{\nabla})\times\mathbf{Z} = \mathbf{Y}\times(\nabla\times\mathbf{Z}) + (\mathbf{Y}\cdot\nabla)\mathbf{Z}$ $-\mathbf{Y}(\nabla\cdot\mathbf{Z})$	$(\mathbf{v}\times\boldsymbol{\nabla})\times\mathbf{A} = \mathbf{v}\times\mathbf{B} + \partial\mathbf{A}/\partial\mathrm{t} - \mathbf{v}(\nabla\cdot\mathbf{A})$. $(\mathbf{v}\times\boldsymbol{\nabla})\times\mathbf{E} = \mathbf{v}\times(\nabla\times\mathbf{E}) + \partial\mathbf{E}/\partial\mathrm{t} - \mathrm{q}\mathbf{v}$. $(\mathbf{v}\times\boldsymbol{\nabla})\times\mathbf{B} = \mathbf{v}\times(\nabla\times\mathbf{B}) + \partial\mathbf{B}/\partial\mathrm{t}$.

5. Electric field and magnetic field created by e-charge moving with acceleration a

(5.1). Divergence of cross product of acceleration a and vector fields: A, E, B

$$\nabla\cdot(\mathbf{Y}\times\mathbf{Z}) = (\nabla\times\mathbf{Y})\cdot\mathbf{Z} - \mathbf{Y}\cdot(\nabla\times\mathbf{Z}) \tag{1}$$

Substituting **(Y** = **a** and **Z** = **A,** vector potential), **(Y** = **a** and **Z** = **E,** electric field**),** and (**Y** = **a** and **Z** = **B,** magnetic field**),** respectively, Equation (1) gives New field equations:

$$\nabla\cdot(\mathbf{a}\times\mathbf{A}) = (\nabla\times\mathbf{a})\cdot\mathbf{A} - \mathbf{a}\cdot(\nabla\times\mathbf{A}) =-\mathbf{a}\cdot\mathbf{B}. \tag{5.1}$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{E}) = (\nabla\times\mathbf{a})\cdot\mathbf{E} - \mathbf{a}\cdot(\nabla\times\mathbf{E}) = \mathbf{a}\cdot(\partial\mathbf{B}/\partial\mathrm{t}). \tag{5.2}$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{B}) = (\nabla\times\mathbf{a})\cdot\mathbf{B} - \mathbf{a}\cdot(\nabla\times\mathbf{B}) =-\mathbf{a}\cdot\mathbf{j} - \mathbf{a}\cdot(\partial\mathbf{E}/\partial\mathrm{t}). \tag{5.3}$$

(5.2). Curl of cross product of acceleration a and vector fields: A, E, B

$$\nabla\times(\mathbf{Y}\times\mathbf{Z}) = \mathbf{Y}(\nabla\cdot\mathbf{Z}) - \mathbf{Z}(\nabla\cdot\mathbf{Y}) + (\mathbf{Z}\cdot\nabla)\mathbf{Y} - (\mathbf{Y}\cdot\nabla)\mathbf{Z} \tag{2}$$

Substituting **(Y** = **a** and **Z** = **A,** vector potential), **(Y** = **a** and **Z** = **E,** electric field), and **(Y** = **a** and **Z** = **B,** magnetic field), respectively, Equation (2) gives NEW field equations:

$$\nabla\times(\mathbf{a}\times\mathbf{A}) = \mathbf{a}(\nabla\cdot\mathbf{A}) - \mathbf{A}(\nabla\cdot\mathbf{a}) + (\mathbf{A}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{A} = \mathbf{a}(\nabla\cdot\mathbf{A}) - (\mathbf{a}\cdot\nabla)\mathbf{A}. \tag{5.4}$$

$$\nabla\times(\mathbf{a}\times\mathbf{E}) = \mathbf{a}(\nabla\cdot\mathbf{E}) - \mathbf{E}(\nabla\cdot\mathbf{a}) + (\mathbf{E}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{E} = \mathrm{q}\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{E}. \tag{5.5}$$

$$\nabla\times(\mathbf{a}\times\mathbf{B}) = \mathbf{a}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{a}) + (\mathbf{B}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{B} =-(\mathbf{a}\cdot\nabla)\mathbf{B}. \tag{5.6}$$

(5.3). Divergence of product of scalar potential φ and acceleration a

$$\nabla \cdot (\varphi \mathbf{Y}) = \varphi(\nabla \cdot \mathbf{Y}) + \mathbf{Y} \cdot (\nabla\varphi) \tag{7}$$

Substituting **Y = a** into Equation (7), we obtain NEW field equation:

$$\nabla \cdot (\varphi \mathbf{a}) = \varphi(\nabla \cdot \mathbf{a}) + \mathbf{a} \cdot (\nabla\varphi) = \mathbf{a} \cdot (\nabla\varphi) \tag{5.7}$$

(5.4) Curl of product of scalar field φ and acceleration a

$$\nabla \times (\varphi \mathbf{Y}) = \varphi(\nabla \times \mathbf{Y}) + (\nabla\varphi) \times \mathbf{Y}. \tag{8}$$

Substituting **Y = a** into Equation (8), we obtain NEW field equation:

$$\nabla \times (\varphi \mathbf{a}) = \varphi(\nabla \times \mathbf{a}) + (\nabla\varphi) \times \mathbf{a} = (\nabla\varphi) \times \mathbf{a} \tag{5.8}$$

(5.5). Gradient of scalar product of acceleration a and vector fields: A, E, B

$$\nabla(\mathbf{Y} \cdot \mathbf{Z}) = (\mathbf{Y} \cdot \nabla)\mathbf{Z} + (\mathbf{Z} \cdot \nabla)\mathbf{Y} + \mathbf{Y} \times (\nabla \times \mathbf{Z}) + \mathbf{Z} \times (\nabla \times \mathbf{Y}). \tag{9}$$

Substituting (**Y** = **a** and **Z** = **A (**vector potential)), (**Y** = **a** and **Z** = **E (**electric field)), and (**Y** = **a** and **Z** = **B (**magnetic field)), into Equation (9), respectively, we obtain NEW field equations:

$$\nabla(\mathbf{a} \cdot \mathbf{A}) = (\mathbf{a} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{a})$$

$$= (\mathbf{a} \cdot \nabla)\mathbf{A} + \mathbf{a} \times \mathbf{B} \tag{5.9}$$

$$\nabla(\mathbf{a} \cdot \mathbf{E}) = (\mathbf{a} \cdot \nabla)\mathbf{E} + (\mathbf{E} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{E}) + \mathbf{E} \times (\nabla \times \mathbf{a})$$

$$= (\mathbf{a} \cdot \nabla)\mathbf{E} + \mathbf{a} \times (\nabla \times \mathbf{E}). \tag{5.10}$$

$$\nabla(\mathbf{a} \cdot \mathbf{B}) = (\mathbf{a} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{a})$$

$$= (\mathbf{a} \cdot \nabla)\mathbf{B} + \mathbf{a} \times (\nabla \times \mathbf{B}). \tag{5.11}$$

(5.6). Cross Product: (Y × ∇) · Z

$$(\mathbf{Y} \times \boldsymbol{\nabla}) \cdot \mathbf{Z} = \mathbf{Y} \cdot (\nabla \times \mathbf{Z}). \tag{10}$$

Substituting **(Y = a** (acceleration) and **Z = A)**, **(Y = a** and **Z = E),** and **(Y = a** and **Z = B),** respectively, Equation (10) gives new field equations,

$$(\mathbf{a} \times \boldsymbol{\nabla}) \cdot \mathbf{A} = \mathbf{a} \cdot (\nabla \times \mathbf{A}) = \mathbf{a} \cdot \mathbf{B}. \tag{5.12}$$

$$(\mathbf{a} \times \boldsymbol{\nabla}) \cdot \mathbf{E} = \mathbf{a} \cdot (\nabla \times \mathbf{E}) =- \mathbf{a} \cdot \partial(\mathbf{B})/\partial \mathrm{t}. \tag{5.13}$$

$$(\mathbf{a} \times \boldsymbol{\nabla}) \cdot \mathbf{B} = \mathbf{a} \cdot (\nabla \times \mathbf{B}) = \mathbf{a} \cdot \mathbf{j} + \mathbf{a} \cdot \partial(\mathbf{E})/\partial \mathrm{t}. \tag{5.14}$$

(5.7). Cross Product: $(\mathbf{Y} \times \boldsymbol{\nabla}) \times \mathbf{Z}$

$$(\mathbf{Y} \times \boldsymbol{\nabla}) \times \mathbf{Z} = \mathbf{Y} \times (\nabla \times \mathbf{Z}) + (\mathbf{Y} \cdot \nabla)\mathbf{Z} - \mathbf{Y}(\nabla \cdot \mathbf{Z}) \tag{11}$$

Substituting **($\mathbf{Y} = \mathbf{a}$** (acceleration) and **$\mathbf{Z} = \mathbf{A}$)**, **($\mathbf{Y} = \mathbf{a}$** and **$\mathbf{Z} = \mathbf{E}$)**, and **($\mathbf{Y} = \mathbf{a}$** and **$\mathbf{Z} = \mathbf{B}$)**, respectively, Equation (11) gives new field equations:

$$(\mathbf{a} \times \boldsymbol{\nabla}) \times \mathbf{A} = \mathbf{a} \times (\nabla \times \mathbf{A}) + (\mathbf{a} \cdot \nabla)\mathbf{A} - \mathbf{a}(\nabla \cdot \mathbf{A})$$

$$= \mathbf{a} \times \mathbf{B} + (\mathbf{a} \cdot \nabla)\mathbf{A} - \mathbf{a}(\nabla \cdot \mathbf{A}). \tag{5.15}$$

$$(\mathbf{a} \times \boldsymbol{\nabla}) \times \mathbf{E} = \mathbf{a} \times (\nabla \times \mathbf{E}) + (\mathbf{a} \cdot \nabla)\mathbf{E} - \mathbf{a}(\nabla \cdot \mathbf{E})$$

$$= \mathbf{a} \times (\nabla \times \mathbf{E}) + (\mathbf{a} \cdot \nabla)\mathbf{E} - \mathrm{q}\mathbf{a}. \tag{5.16}$$

$$(\mathbf{a} \times \boldsymbol{\nabla}) \times \mathbf{B} = \mathbf{a} \times (\nabla \times \mathbf{B}) + (\mathbf{a} \cdot \nabla)\mathbf{B} - \mathbf{a}(\nabla \cdot \mathbf{B})$$

$$= \mathbf{a} \times (\nabla \times \mathbf{B}) + (\mathbf{a} \cdot \nabla)\mathbf{B}. \tag{5.17}$$

Summary of Section 5: Electric field and magnetic field created by accelerating electric charge

Vector Calculus Identities	**a, A, E** and **B**
$\nabla \cdot (\mathbf{Y} \times \mathbf{Z}) = (\nabla \times \mathbf{Y}) \cdot \mathbf{Z} - \mathbf{Y} \cdot (\nabla \times \mathbf{Z})$	$\nabla \cdot (\mathbf{a} \times \mathbf{A}) = -\mathbf{a} \cdot \mathbf{B}$. $\nabla \cdot (\mathbf{a} \times \mathbf{E}) = \mathbf{a} \cdot (\partial \mathbf{B}/\partial \mathrm{t})$. $\nabla \cdot (\mathbf{a} \times \mathbf{B}) = -\mathbf{a} \cdot \mathbf{j} - \mathbf{a} \cdot (\partial \mathbf{E}/\partial \mathrm{t})$.
$\nabla \times (\mathbf{Y} \times \mathbf{Z}) = \mathbf{Y}(\nabla \cdot \mathbf{Z}) - \mathbf{Z}(\nabla \cdot \mathbf{Y})$ $+ (\mathbf{Z} \cdot \nabla)\mathbf{Y} - (\mathbf{Y} \cdot \nabla)\mathbf{Z}$	$\nabla \times (\mathbf{a} \times \mathbf{A}) = \mathbf{a}(\nabla \cdot \mathbf{A}) - (\mathbf{a} \cdot \nabla)\mathbf{A}$. $\nabla \times (\mathbf{a} \times \mathbf{E}) = \mathrm{q}\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{E}$. $\nabla \times (\mathbf{a} \times \mathbf{B}) = -(\mathbf{a} \cdot \nabla)\mathbf{B}$.
$\nabla \cdot (\varphi \mathbf{Y}) = \varphi(\nabla \cdot \mathbf{Y}) + \mathbf{Y} \cdot (\nabla \varphi)$	$\nabla \cdot (\varphi \mathbf{a}) = \mathbf{a} \cdot (\nabla \varphi)$
$\nabla \times (\varphi \mathbf{Y}) = \varphi(\nabla \times \mathbf{Y}) + (\nabla \varphi) \times \mathbf{Y}$	$\nabla \times (\varphi \mathbf{a}) = (\nabla \varphi) \times \mathbf{a}$
$\nabla(\mathbf{Y} \cdot \mathbf{Z}) = (\mathbf{Y} \cdot \nabla)\mathbf{Z} + (\mathbf{Z} \cdot \nabla)\mathbf{Y}$ $+\mathbf{Y} \times (\nabla \times \mathbf{Z}) + \mathbf{Z} \times (\nabla \times \mathbf{Y})$	$\nabla(\mathbf{a} \cdot \mathbf{A}) = (\mathbf{a} \cdot \nabla)\mathbf{A} + \mathbf{a} \times \mathbf{B}$ $\nabla(\mathbf{a} \cdot \mathbf{E}) = (\mathbf{a} \cdot \nabla)\mathbf{E} + \mathbf{a} \times (\nabla \times \mathbf{E})$. $\nabla(\mathbf{a} \cdot \mathbf{B}) = (\mathbf{a} \cdot \nabla)\mathbf{B} + \mathbf{a} \times (\nabla \times \mathbf{B})$.
$(\mathbf{Y} \times \boldsymbol{\nabla}) \cdot \mathbf{Z} = \mathbf{Y} \cdot (\nabla \times \mathbf{Z})$	$(\mathbf{a} \times \boldsymbol{\nabla}) \cdot \mathbf{A} = \mathbf{a} \cdot \mathbf{B}$. $(\mathbf{a} \times \boldsymbol{\nabla}) \cdot \mathbf{E} = -\mathbf{v} \cdot \partial \mathbf{B}/\partial \mathrm{t}$. $(\mathbf{a} \times \boldsymbol{\nabla}) \cdot \mathbf{B} = \mathbf{a} \cdot \mathbf{j} + \mathbf{a} \cdot \partial(\mathbf{E})/\partial \mathrm{t}$.
$(\mathbf{Y} \times \boldsymbol{\nabla}) \times \mathbf{Z} = \mathbf{Y} \times (\nabla \times \mathbf{Z}) + (\mathbf{Y} \cdot \nabla)\mathbf{Z}$	$(\mathbf{a} \times \boldsymbol{\nabla}) \times \mathbf{A} = \mathbf{a} \times \mathbf{B} + (\mathbf{a} \cdot \nabla)\mathbf{A} - \mathbf{a}(\nabla \cdot \mathbf{A})$.

$-\mathbf{Y}(\nabla\cdot\mathbf{Z})$	$(\mathbf{a}\times\boldsymbol{\nabla})\times\mathbf{E}=\mathbf{a}\times(\nabla\times\mathbf{E})+(\mathbf{a}\cdot\nabla)\mathbf{E}-\mathrm{q}\mathbf{a}$ $(\mathbf{a}\times\boldsymbol{\nabla})\times\mathbf{B}=\mathbf{a}\times(\nabla\times\mathbf{B})+(\mathbf{a}\cdot\nabla)\mathbf{B}.$

6. Electric field E and magnetic field B created by electric charge with spin S

(6.1). Divergence of cross product of Spin and vector field: $\mathbf{S}\times\mathbf{Z}$

$$\nabla\cdot(\mathbf{Y}\times\mathbf{Z})=(\nabla\times\mathbf{Y})\cdot\mathbf{Z}-\mathbf{Y}\cdot(\nabla\times\mathbf{Z}) \quad (1)$$

Substituting **(Y** = **S** (Spin), **Z** = **A (**vector potential)), **(Y** = **S** (Spin), **Z** = **E** (electric field)), **(Y** = **S** (Spin), **Z** = **B** (magnetic field)), respectively, Equation (1) gives New field equation

$$\nabla\cdot(\mathbf{S}\times\mathbf{A})=(\nabla\times\mathbf{S})\cdot\mathbf{A}-\mathbf{S}\cdot(\nabla\times\mathbf{A})=-\mathbf{S}\cdot\mathbf{B}. \quad (6.1)$$

$$\nabla\cdot(\mathbf{S}\times\mathbf{E})=(\nabla\times\mathbf{S})\cdot\mathbf{E}-\mathbf{S}\cdot(\nabla\times\mathbf{E})=\mathbf{S}\cdot\partial(\mathbf{B})/\partial\mathrm{t}. \quad (6.2)$$

$$\nabla\cdot(\mathbf{S}\times\mathbf{B})=(\nabla\times\mathbf{S})\cdot\mathbf{B}-\mathbf{S}\cdot(\nabla\times\mathbf{B})=-\mathbf{S}\cdot\mathbf{j}-\mathbf{S}\cdot\partial(\mathbf{E})/\partial\mathrm{t} \quad (6.3)$$

(6.2). Curl of cross product of Spin and vector fields: $\mathbf{S}\times\mathbf{Z}$

Curl of cross product of Spin S and vector fields: $\mathbf{S},\mathbf{Z}$

$$\nabla\times(\mathbf{Y}\times\mathbf{Z})=\mathbf{Y}(\nabla\cdot\mathbf{Z})-\mathbf{Z}(\nabla\cdot\mathbf{Y})+(\mathbf{Z}\cdot\nabla)\mathbf{Y}-(\mathbf{Y}\cdot\nabla)\mathbf{Z} \quad (2)$$

Substituting **(Y** = **S** and **Z** = **A), (Y** = **S** and **Z** = **E),** and **(Y** = **S** and **Z** = **B),** respectively, Equation (2) gives NEW field equations:

$$\nabla\times(\mathbf{S}\times\mathbf{A})=\mathbf{S}(\nabla\cdot\mathbf{A})-\mathbf{A}(\nabla\cdot\mathbf{S})+(\mathbf{A}\cdot\nabla)\mathbf{S}-(\mathbf{S}\cdot\nabla)\mathbf{A}=\mathbf{S}(\nabla\cdot\mathbf{A})-(\mathbf{S}\cdot\nabla)\mathbf{A}. \quad (6.4)$$

$$\nabla\times(\mathbf{S}\times\mathbf{E})=\mathbf{S}(\nabla\cdot\mathbf{E})-\mathbf{E}(\nabla\cdot\mathbf{S})+(\mathbf{E}\cdot\nabla)\mathbf{S}-(\mathbf{S}\cdot\nabla)\mathbf{E}=\mathrm{q}\mathbf{S}-(\mathbf{S}\cdot\nabla)\mathbf{E}. \quad (6.5)$$

$$\nabla\times(\mathbf{S}\times\mathbf{B})=\mathbf{S}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{S})+(\mathbf{B}\cdot\nabla)\mathbf{S}-(\mathbf{S}\cdot\nabla)\mathbf{B}=-(\mathbf{S}\cdot\nabla)\mathbf{B}. \quad (6.6)$$

(6.3). Divergence of product of scalar potential φ and Spin S

$$\nabla\cdot(\varphi\mathbf{Y})=\varphi(\nabla\cdot\mathbf{Y})+\mathbf{Y}\cdot(\nabla\varphi) \quad (7)$$

Substituting **Y = S** (Spin) into Equation (7), we obtain NEW field equations:

$$\nabla\cdot(\varphi\mathbf{S})=\varphi(\nabla\cdot\mathbf{S})+\mathbf{S}\cdot(\nabla\varphi)=\mathbf{S}\cdot(\nabla\varphi) \quad (6.7)$$

(6.4) Curl of product of scalar field φ and Spin S

$$\nabla \times (\varphi \mathbf{Y}) = \varphi(\nabla \times \mathbf{Y}) + (\nabla\varphi) \times \mathbf{Y}. \tag{8}$$

Substituting **Y = S** into Equation (8), we obtain NEW field equations:

$$\nabla \times (\varphi \mathbf{S}) = \varphi(\nabla \times \mathbf{S}) + (\nabla\varphi) \times \mathbf{S} = (\nabla\varphi) \times \mathbf{S} \tag{6.8}$$

(6.5). Gradient of scalar product of Spin and vector fields: $\nabla(\mathbf{S} \cdot \mathbf{A})$, $\nabla(\mathbf{S} \cdot \mathbf{E})$, and $\nabla(\mathbf{S} \cdot \mathbf{B})$

$$\nabla(\mathbf{Y} \cdot \mathbf{Z}) = (\mathbf{Y} \cdot \nabla)\mathbf{Z} + (\mathbf{Z} \cdot \nabla)\mathbf{Y} + \mathbf{Y} \times (\nabla \times \mathbf{Z}) + \mathbf{Z} \times (\nabla \times \mathbf{Y}). \tag{9}$$

Substituting ($\mathbf{Y} = \mathbf{S}$ and $\mathbf{Z} = \mathbf{A}$), ($\mathbf{Y} = \mathbf{S}$ and $\mathbf{Z} = \mathbf{E}$), and ($\mathbf{Y} = \mathbf{S}$ and $\mathbf{Z} = \mathbf{B}$), into Equation (9), respectively, we obtain **NEW** field equation:

$$\begin{aligned}\nabla(\mathbf{S} \cdot \mathbf{A}) &= (\mathbf{S} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{S} + \mathbf{S} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{S}) \\ &= (\mathbf{S} \cdot \nabla)\mathbf{A} + \mathbf{S} \times \mathbf{B}\end{aligned} \tag{6.9}$$

$$\begin{aligned}\nabla(\mathbf{S} \cdot \mathbf{E}) &= (\mathbf{S} \cdot \nabla)\mathbf{E} + (\mathbf{E} \cdot \nabla)\mathbf{S} + \mathbf{S} \times (\nabla \times \mathbf{E}) + \mathbf{E} \times (\nabla \times \mathbf{S}) \\ &= (\mathbf{S} \cdot \nabla)\mathbf{E} + \mathbf{S} \times (\nabla \times \mathbf{E}).\end{aligned} \tag{6.10}$$

$$\begin{aligned}\nabla(\mathbf{S} \cdot \mathbf{B}) &= (\mathbf{S} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{S} + \mathbf{S} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{S}) \\ &= (\mathbf{S} \cdot \nabla)\mathbf{B} + \mathbf{S} \times (\nabla \times \mathbf{B}).\end{aligned} \tag{6.11}$$

(6.6). Cross Product: $(\mathbf{Y} \times \boldsymbol{\nabla}) \cdot \mathbf{Z}$

$$(\mathbf{Y} \times \boldsymbol{\nabla}) \cdot \mathbf{Z} = \mathbf{Y} \cdot (\nabla \times \mathbf{Z}). \tag{10}$$

Substituting **($\mathbf{Y} = \mathbf{S}$ and $\mathbf{Z} = \mathbf{A}$)**, **($\mathbf{Y} = \mathbf{S}$ and $\mathbf{Z} = \mathbf{E}$),** and **($\mathbf{Y} = \mathbf{S}$ and $\mathbf{Z} = \mathbf{B}$),** respectively, Equation (10) gives new field equation,

$$(\mathbf{S} \times \boldsymbol{\nabla}) \cdot \mathbf{A} = \mathbf{S} \cdot (\nabla \times \mathbf{A}) = \mathbf{S} \cdot \mathbf{B}. \tag{6.12}$$

$$(\mathbf{S} \times \boldsymbol{\nabla}) \cdot \mathbf{E} = \mathbf{S} \cdot (\nabla \times \mathbf{E}) = - \mathbf{S} \cdot \partial(\mathbf{B})/\partial t. \tag{6.13}$$

$$(\mathbf{S} \times \boldsymbol{\nabla}) \cdot \mathbf{B} = \mathbf{S} \cdot (\nabla \times \mathbf{B}) = \mathbf{S} \cdot \mathbf{j} + \mathbf{S} \cdot \partial(\mathbf{E})/\partial t. \tag{6.14}$$

(6.7). Cross Product: $(\mathbf{Y} \times \boldsymbol{\nabla}) \times \mathbf{Z}$

$$(\mathbf{Y} \times \boldsymbol{\nabla}) \times \mathbf{Z} = \mathbf{Y} \times (\nabla \times \mathbf{Z}) + (\mathbf{Y} \cdot \nabla)\mathbf{Z} - \mathbf{Y}(\nabla \cdot \mathbf{Z}) \tag{11}$$

Substituting **($\mathbf{Y} = \mathbf{S}$ and $\mathbf{Z} = \mathbf{A}$)**, **($\mathbf{Y} = \mathbf{S}$ and $\mathbf{Z} = \mathbf{E}$)**, and **($\mathbf{Y} = \mathbf{S}$ and $\mathbf{Z} = \mathbf{B}$)**, respectively, Equation (11) gives new field equations:

$$(\mathbf{S} \times \boldsymbol{\nabla}) \times \mathbf{A} = \mathbf{S} \times (\nabla \times \mathbf{A}) + (\mathbf{S} \cdot \nabla)\mathbf{A} - \mathbf{S}(\nabla \cdot \mathbf{A})$$

$$= \mathbf{S} \times \mathbf{B} + (\mathbf{S} \cdot \nabla)\mathbf{A} - \mathbf{S}(\nabla \cdot \mathbf{A}). \tag{6.15}$$

$$(\mathbf{S} \times \boldsymbol{\nabla}) \times \mathbf{E} = \mathbf{S} \times (\nabla \times \mathbf{E}) + (\mathbf{S} \cdot \nabla)\mathbf{E} - \mathbf{S}(\nabla \cdot \mathbf{E})$$

$$= \mathbf{S} \times (\nabla \times \mathbf{E}) + (\mathbf{S} \cdot \nabla)\mathbf{E} - \mathrm{q}\mathbf{S}. \tag{6.16}$$

$$(\mathbf{S} \times \boldsymbol{\nabla}) \times \mathbf{B} = \mathbf{S} \times (\nabla \times \mathbf{B}) + (\mathbf{S} \cdot \nabla)\mathbf{B} - \mathbf{S}(\nabla \cdot \mathbf{B})$$

$$= \mathbf{S} \times (\nabla \times \mathbf{B}) + (\mathbf{S} \cdot \nabla)\mathbf{B}. \tag{6.17}$$

Summary of Section 6: Electric field and magnetic field created by Spin electric charge

Vector Calculus Identities	**S, A, E** and **B**
$\nabla \cdot (\mathbf{Y} \times \mathbf{Z}) = (\nabla \times \mathbf{Y}) \cdot \mathbf{Z} - \mathbf{Y} \cdot (\nabla \times \mathbf{Z})$	$\nabla \cdot (\mathbf{S} \times \mathbf{A}) =- \mathbf{S} \cdot \mathbf{B}..$ $\nabla \cdot (\mathbf{S} \times \mathbf{E}) = \mathbf{S} \cdot \partial(\mathbf{B})/\partial \mathbf{t}.$ $\nabla \cdot (\mathbf{S} \times \mathbf{B}) =- \mathbf{S} \cdot \mathbf{j} - \mathbf{S} \cdot \partial(\mathbf{E})/\partial \mathrm{t}$
$\nabla \times (\mathbf{Y} \times \mathbf{Z}) = \mathbf{Y}(\nabla \cdot \mathbf{Z}) - \mathbf{Z}(\nabla \cdot \mathbf{Y})$ $+ (\mathbf{Z} \cdot \nabla)\mathbf{Y} - (\mathbf{Y} \cdot \nabla)\mathbf{Z}$	$\nabla \times (\mathbf{S} \times \mathbf{A}) = \mathbf{S}(\nabla \cdot \mathbf{A}) - (\mathbf{S} \cdot \nabla)\mathbf{A}.$ $\nabla \times (\mathbf{S} \times \mathbf{E}) = \mathrm{q}\mathbf{S} - (\mathbf{S} \cdot \nabla)\mathbf{E}.$ $\nabla \times (\mathbf{S} \times \mathbf{B}) =- (\mathbf{S} \cdot \nabla)\mathbf{B}$
$\nabla \cdot (\varphi\mathbf{Y}) = \varphi(\nabla \cdot \mathbf{Y}) + \mathbf{Y} \cdot (\nabla\varphi)$	$\nabla \cdot (\varphi\mathbf{S}) = \mathbf{S} \cdot (\nabla\varphi)$
$\nabla \times (\varphi\mathbf{Y}) = \varphi(\nabla \times \mathbf{Y}) + (\nabla\varphi) \times \mathbf{Y}$	$\nabla \times (\varphi\mathbf{S}) = (\nabla\varphi) \times \mathbf{S}$
$\nabla(\mathbf{Y} \cdot \mathbf{Z}) = (\mathbf{Y} \cdot \nabla)\mathbf{Z} + (\mathbf{Z} \cdot \nabla)\mathbf{Y}$ $+\mathbf{Y} \times (\nabla \times \mathbf{Z}) + \mathbf{Z} \times (\nabla \times \mathbf{Y})$	$\nabla(\mathbf{S} \cdot \mathbf{A}) = (\mathbf{S} \cdot \nabla)\mathbf{A} + \mathbf{S} \times \mathbf{B}.$ $\nabla(\mathbf{S} \cdot \mathbf{E}) = (\mathbf{S} \cdot \nabla)\mathbf{E} + \mathbf{S} \times (\nabla \times \mathbf{E})$ $\nabla(\mathbf{S} \cdot \mathbf{B}) = (\mathbf{S} \cdot \nabla)\mathbf{B} + \mathbf{S} \times (\nabla \times \mathbf{B}).$
$(\mathbf{Y} \times \boldsymbol{\nabla}) \cdot \mathbf{Z} = \mathbf{Y} \cdot (\nabla \times \mathbf{Z})$	$(\mathbf{S} \times \boldsymbol{\nabla}) \cdot \mathbf{A} = \mathbf{S} \cdot (\nabla \times \mathbf{A}) = \mathbf{S} \cdot \mathbf{B}.$ $(\mathbf{S} \times \boldsymbol{\nabla}) \cdot \mathbf{E} =- \mathbf{S} \cdot \partial(\mathbf{B})/\partial \mathrm{t}.$ $(\mathbf{S} \times \boldsymbol{\nabla}) \cdot \mathbf{B} = \mathbf{S} \cdot \mathbf{j} + \mathbf{S} \cdot \partial(\mathbf{E})/\partial \mathrm{t}.$
$(\mathbf{Y} \times \boldsymbol{\nabla}) \times \mathbf{Z} = \mathbf{Y} \times (\nabla \times \mathbf{Z}) + (\mathbf{Y} \cdot \nabla)\mathbf{Z}$ $-\mathbf{Y}(\nabla \cdot \mathbf{Z})$	$(\mathbf{S} \times \boldsymbol{\nabla}) \times \mathbf{A} = \mathbf{S} \times \mathbf{B} + (\mathbf{S} \cdot \nabla)\mathbf{A} - \mathbf{S}(\nabla \cdot \mathbf{A}).$ $(\mathbf{S} \times \boldsymbol{\nabla}) \times \mathbf{E} = \mathbf{S} \times (\nabla \times \mathbf{E}) + (\mathbf{S} \cdot \nabla)\mathbf{E} - \mathrm{q}\mathbf{S}.$ $(\mathbf{S} \times \boldsymbol{\nabla}) \times \mathbf{B} = \mathbf{S} \times (\nabla \times \mathbf{B}) + (\mathbf{S} \cdot \nabla)\mathbf{B}.$

7. Summery

In this article, we show that the vector calculus identities are powerful tool in re-discovering existing physical laws, and in discovering novel physical laws. We rederived mathematically the following existing physical laws by the vector calculus identities:

(1). Poynting's Theorem:

$$\nabla\cdot(\mathbf{E}\times\mathbf{B})=\left(\nabla\times\mathbf{E}\right)\cdot\mathbf{B}-\mathbf{E}\cdot\left(\nabla\times\mathbf{B}\right)=-\tfrac{1}{2}\partial(\mathbf{E}^2+\mathbf{B}^2)/\partial t-\mathbf{E}\cdot\mathbf{j}$$

(2). Gauss's Law for Magnetism:

$$\nabla\cdot\left(\nabla\times\mathbf{A}\right)=\nabla\cdot\mathbf{B}=0$$

$$\nabla\cdot\left(\nabla\times\mathbf{E}\right)=-\nabla\cdot\frac{\partial\mathbf{B}}{\partial t}=-\frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{B}\right)=0.$$

(3). Charge density **C**ontinuity equation**:**

$$\nabla\cdot\left(\nabla\times\mathbf{B}\right)=\nabla\cdot\mathbf{j}+\nabla\cdot\frac{\partial\mathbf{E}}{\partial t}=\nabla\cdot\mathbf{j}+\frac{\partial}{\partial t}\rho=0$$

(4). Wave equations of potentials φ and **A**, electric field **E**, magnetic field **B**:

$$\frac{\partial^2}{\partial t^2}\mathbf{A}-\nabla^2\mathbf{A}=\mathbf{j},$$

$$\frac{\partial^2}{\partial t^2}\mathbf{E}-\nabla^2\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{j}-\nabla\rho,$$

$$\frac{\partial^2}{\partial t^2}\mathbf{B}-\nabla^2\mathbf{B}=\nabla\times\mathbf{j}$$

$$\frac{\partial^2}{\partial t^2}\varphi-\nabla^2\varphi=\mathrm{q}.$$

(5). Faraday's Law of Induction:

$$\nabla\times\left(\nabla\varphi\right)=\nabla\times\left(-\mathbf{E}-\frac{\partial\mathbf{A}}{\partial \mathrm{t}}\right)=-\nabla\times\mathbf{E}-\frac{\partial\mathbf{B}}{\partial \mathrm{t}}=0.$$

(6). Time changes of energy density of the magnetic field:

$$(\mathbf{B}\times\boldsymbol{\nabla})\cdot\mathbf{E}=\mathbf{B}\cdot\left(\nabla\times\mathbf{E}\right)=-\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}.$$

(7). Time changes of energy density of the electric field;

$$(\mathbf{E}\times\boldsymbol{\nabla})\cdot\mathbf{B}=\mathbf{E}\cdot\left(\nabla\times\mathbf{B}\right)=\mathbf{E}\cdot\mathbf{j}+\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}$$

(8). Energy density of the magnetic field:

$$(\mathbf{B} \times \boldsymbol{\nabla}) \cdot \mathbf{A} = \mathbf{B} \cdot (\nabla \times \mathbf{A}) = \mathbf{B}^2.$$

Also, in this article, applying the vector calculus identities, we further derived/proposed more than 50 new field equations of both electric fields and magnetic fields. The new field equations need to be tested experimentally.

Reference

[1] Vector calculus identities, Wikipedia.