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A suggested exact numerical value for the fine-structure constant

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Abstract:

The effect of extending from 10 to 14 digits the precision of the numerical value of α , the Sommerfeld constant, is shown to be effective in generating equations to define and calculate matching numeric values of several important α -related fundamental physical constants instituted as exact. It is also demonstrated that the electron mass and the proton mass numerical values published in CODATA 2018 inventory ought to be used to properly compound the above α tweak to further develop new equations; specifically, some electron to mass quotient expressions, a set of exact h expressions with two showing its main fundamental energy/frequency physical interpretation and a couple of α physical interpretations from using the Rydberg photon mass and the Bohr mini magneton concepts. A straightforward calculation procedure to derive the proton charge radius is disclosed. An α physical interpretation based on two entropy entities is conferred. A group of expressions to calculate the intergalactic hydrogen line H_{21} (m) and the proton-electron mass ratio are disclosed.

Keywords: Sommerfeld constant, exact expressions for h , c , and e , the Planck mini constants, the Rydberg photon mass, the Bohr mini magneton, the electron-mass quotient, the proton charge radius, the hydrogen 21 cm wavelength, the proton-electron mass ratio, the c/a_0 frequency.

The extended Sommerfeld constant definition and interpretation.

The exact constants h , c and e expressions.

On what follows, we'll make use of the numeric value of α , the Sommerfeld or fine-structure constant, calculated using the physical interpretation definition given in [NIST \$\alpha\$](#) , renamed as α_e and extend its numeric value to 14 digits as follows

$$\alpha_e = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{e^2}{q_p^2} = 7.297\,352\,5643\,3301 \times 10^{-3} \approx^{\text{NIST}} 7.297\,352\,5643(11) \times 10^{-3} \quad (1)$$

where the numeric value and uncertainty at the right is as currently given in above link and e (C), ϵ_0 (F/m), $\hbar$ (J·s) and c (m·s⁻¹) stand for the electron charge, the vacuum electric permittivity, the reduced Planck quantum unit of action constant and the speed of light in vacuum, respectively. q_p (C) is the Planck quantum charge. The relative shift among the two numeric values above is $1 - \alpha/\alpha_e = 4.563 \times 10^{-11}$. Then, (1) states that this fine-structure constant physical interpretation is as the ratio of a particle-like classical mechanics entity divided by a wave-like quantum mechanics entity.

From (1) the following physical interpretation as the ratio of two entities with unit of action units (J·s), the numerator one being smaller than h , say $h_{m,1}$ or a sort of Planck mini unit of action, can be derived as follows

$$\alpha_e = \frac{e^2/(2\epsilon_0 c)}{h} = \frac{h_{m,1}}{h} \quad (2)$$

Solving (2) for h and calculating gives its exact value as follows

$$h = \frac{e^2}{2\epsilon_0\alpha_e c} = 6.626\,070\,15 \times 10^{-34} \text{ (exact)} \left(\text{C}^2/\frac{\text{F}}{\text{s}} \right) \text{ or } (\text{J} \cdot \text{s}) \quad (3)$$

Note that (3) does not supply an intuitive h physical interpretation, namely energy/frequency. We'll return to this topic further down.

Now, solving (3) for c gives

$$c = \frac{e^2}{2\epsilon_0\alpha_e h} = 299\,792\,458 \text{ (exact)} \text{ (m} \cdot \text{s}^{-1}) \quad (4)$$

Solving this for e leads to

$$e = -\sqrt{2\epsilon_0\alpha_e h c} = -1.602\,176\,634 \times 10^{-19} \text{ (exact)} \text{ (C)} \quad (5)$$

Finally, solving for ϵ_0 provides

$$\epsilon_0 = \frac{e^2}{2\alpha_e h c} = 8.854\,187\,818\,800\,00 \times 10^{-12} \text{ (F} \cdot \text{m}^{-1}) \quad (6)$$

which gives an all-digit match to the NIST numeric value [vcp](#) $8.854\,187\,818\,8(19) \times 10^{-12}$ (F/m). With (4) and (7), the extended numeric value of the vacuum magnetic permeability can be calculated as follows

$$\mu_{0,e} = \frac{1}{\epsilon_0 c^2} = 1.256\,637\,061\,2685 \times 10^{-6} \approx^{\text{NIST}} 1.256\,637\,061\,27(19) \times 10^{-6} \left(\frac{\text{N}}{\text{A}^2} \right) \quad (7)$$

which gives also a 10-digit match to the NIST numeric value [Permeability](#) and, its provided definition gives also

$$\mu_{0,e} = \frac{2\alpha_e h}{e^2 c} = 1.256\,637\,061\,2685 \times 10^{-6} \left(\frac{\text{N}}{\text{A}^2} \right) \quad (8)$$

The relative shift among the numeric values in (7) is $1 - \mu_{0,e}/\mu_0 = 1.194 \times 10^{-12}$.

The electron mass numeric value. The energy/frequency interpretations of h .

Addressing other subject, for the electron mass or atomic unit of mass numeric value, we'll make use of the one given in CODATA 2018 as $m_{e,18}$ [me2018](#) (kg) instead of the current $m_{e,22}$ (kg) [aum](#) which differ as shown here

$$m_{e,18} = 9.109\,383\,7015 \times 10^{-31} \text{ (kg)} \quad (9)$$

$$m_{e,22} = 9.109\,383\,7139 \times 10^{-31} \text{ (kg)} \quad (10)$$

where their relative numeric value shift is $1 - m_{e,18}/m_{e,22} = 1.361 \times 10^{-9}$.

Let's now elaborate on (3) as follows

$$h = \frac{e^2}{2\varepsilon_0 \alpha_e c} = \frac{e^2}{2\varepsilon_0 v_{1,e}} \frac{a_{0,e}^2}{a_{0,e}^2} \frac{2\pi}{2\pi} = \left\{ \frac{1}{v_{1,e}/2\pi a_{0,e}} \right\} a_{0,e} \left\{ \frac{e^2}{4\pi\varepsilon_0 a_{0,e}^2} \right\} \text{ (J} \cdot \text{s)} \quad (11)$$

where $v_{1,e}$ (m/s) stands for the extended electron first orbit speed in the Bohr H model also known as the atomic unit of velocity whose numeric value, using α_e as given in (1), is

$$v_{1,e} = \alpha_e c = 2.187\,691\,262\,154 \times 10^6 \approx^{\text{NIST}} 2.187\,691\,262\,16(34) \times 10^6 \text{ (m/s)} \quad (12)$$

where their relative shift is $1 - v_{1,e}/v_1 = 2.744 \times 10^{-12}$. $a_{0,e}$ (m) is the H atom radius or Bohr radius or atomic unit of length [aul](#) whose numeric value calculated using NIST equation and above given data is

$$a_{0,e} = \frac{\hbar}{(\alpha_e m_{e,18}) c} = 5.291\,772\,112\,6468 \times 10^{-11} \approx^{\text{NIST}} 5.291\,772\,105\,44(82) \times 10^{-11} \text{ (m)} \quad (13)$$

where the implied relative shift is $1 - a_0/a_{0,e} = 1.362 \times 10^{-9}$. Then,

$$h = \frac{a_{0,e} F_{C,e}}{\varphi_1} = 6.626\,070\,150\,000\,000 \times 10^{-34} \left(\frac{\text{m} \cdot \text{N}}{\text{Hz}} \text{ or } \frac{\text{J}}{\text{Hz}} \right) \quad (14)$$

where the following definition was introduced

$$\varphi_1 = v_{1,e}/2\pi a_{0,e} = 6.579\,683\,911\,531\,950 \times 10^{15} \text{ (Hz or 1/s)} \quad (15)$$

which represents the electron frequency or number of turns around the H atom diameter per second. $F_{C,e}$ is the magnitude of the Coulomb or electric force between the proton and the electron of the H atom also known as the atomic unit of force [auf](#) and, using above defined numeric values, is given as

$$F_{C,e} = \frac{e^2}{4\pi\epsilon_0 a_{0,e}^2} = 8.238\,723\,481\,391\,730 \times 10^8 \approx^{\text{NIST}} 8.238\,723\,5038(13) \times 10^8 \text{ (N)} \quad (16)$$

where the relative shift is $1 - F_{C,e}/F_C = 2.72 \times 10^{-9}$. Then, (14) gives a physical interpretation of h as a moment of a force or torque or energy per each electron cycle.

Using (1) and (13) in the electron classical radius definition given in [ecr](#) yields

$$r_{e,e} = a_{0,e} \alpha_e^2 = 2.817\,940\,324\,295\,36 \times 10^{-15} \approx^{\text{NIST}} 2.817\,940\,3205(13) \times 10^{-15} \text{ (m)} \quad (17)$$

whose relative shift is $1 - r_e/r_{e,e} = 1.347 \times 10^{-9}$.

Using (9), (12), (13) and (17), provides the following h exact expressions

$$h = 2\pi\alpha_e a_{0,e} m_{e,18} c = 2\pi a_{0,e} m_{e,18} v_{1,e} = 2\pi r_{e,e} m_{e,18} c / \alpha_e = 2\pi r_{e,e} m_{e,18} c^2 / v_{1,e} \text{ (J} \cdot \text{s)} \quad (18)$$

The last term can be transformed into another fundamental physical interpretation of h by writing it as

$$h = \frac{E_{e,e}}{\varphi_2} \left(\frac{\text{J}}{\text{Hz}} \right) \quad (19)$$

where $E_{e,e}$ (J) is the electron mass-energy equivalent given by

$$E_{e,e} = m_{e,18} c^2 = 8.187\,105\,776\,823\,39 \times 10^{-14} \text{ (J)} \quad (20)$$

and φ_2 stands for the frequency or number of times the electron travels the electron perimeter $2\pi r_{e,e}$ (cm) distance in a second and is defined as

$$\varphi_2 = v_{1,e} / 2\pi r_{e,e} = 1.235\,589\,963\,807\,410 \times 10^{20} \text{ (Hz or 1/s)} \quad (21)$$

The Rydberg constant

Let's now consider $R_{\infty,e}$ (1/m), the Rydberg constant, widely used in the emission spectral series of the Bohr H model to represent the invers of the shortest possible wavelength $\lambda_{\infty,e}$ (m) in the Lyman series.

$$R_{\infty,e} = \frac{\alpha_e^2 m_{e,18} c}{2h} = \frac{r_{e,e} m_{e,18} c}{2a_{0,e} h} = \frac{m_{e,18} e^4}{8\epsilon_0^2 c h^3} = \frac{\alpha_e}{4\pi a_{0,e}} = \frac{1}{\lambda_{\infty,e}} = 10\,973\,731.553\,200 \text{ (m}^{-1}\text{)} \quad (22)$$

where all five expressions give the same numeric value. The NIST R_{∞} numerical value [Ryc](#) is

$$R_{\infty} = 10\,973\,731.568\,157(12) \text{ (m}^{-1}\text{)} \quad (23)$$

The relative shift between (22) and (23) is $1 - R_{\infty,e}/R_{\infty} = 1.363 \times 10^{-9}$.

From the last two expressions in (22), it follows that

$$\lambda_{\infty,e} = \frac{4\pi a_{0,e}}{\alpha_e} = 9.112\,670\,518\,246\,80 \times 10^{-8} \text{ (m)} \quad (24)$$

while $\lambda_{\infty} = 1/R_{\infty}$ (m) is

$$\lambda_{\infty} = \frac{4\pi a_0}{\alpha} = 9.112\ 670\ 505\ 826\ 37 \times 10^{-8} \text{ (m)} \quad (25)$$

where the relative shift is $1 - \lambda_{\infty}/\lambda_{\infty,e} = 1.363 \times 10^{-9}$ which coincides, inversely, with the previous one.

Using the Compton wavelength concept of a particle mass but, instead, used to determine the mass of a photon which has the Rydberg wavelength, $\lambda_{\infty,e}$ is given as

$$m_{e,Ry} = 2h/(\lambda_{\infty,e}c) = 4.850\ 870\ 202\ 930\ 29 \times 10^{-35} \text{ (kg)} \quad (26)$$

which allows us to write the following fine-structure constant physical interpretation, also showing the wave-particle duality, as follows

$$\alpha_e = \left(\frac{m_{e,Ry}}{m_{e,18}} \right)^{1/2} = 7.297\ 352\ 5643\ \mathbf{3301} \times 10^{-3} \quad (27)$$

The H atom ionization energy is calculated as follows

$$E_{Ry,e} = \frac{hc}{\lambda_{\infty,e}} = 2.179\ 872\ 358\ \mathbf{131\ 86} \times 10^{-18} \text{ (J)} \quad (28)$$

while the NIST numerical value is

$$E_{Ry} = \frac{hc}{\lambda_{\infty}} = 2.179\ 872\ 361\ 1030(24) \times 10^{-18} \text{ (J)} \quad (29)$$

where the relative shift is $1 - E_{Ry,e}/E_{Ry} = 1.363 \times 10^{-9}$ as the two previous ones.

Let's transform (28) into

$$hc = E_{Ry,e}\lambda_{\infty,e} = 1.986\ 445\ 857\ 148\ 930 \times 10^{-25} \text{ (J} \cdot \text{m)} \quad (30)$$

both terms are numerically identical. The same process in (29) gives

$$hc = E_{Ry}\lambda_{\infty} = 1.986\ 445\ 857\ 148\ 930 \times 10^{-25} \text{ (J} \cdot \text{m)} \quad (31)$$

both terms are also numerically identical and, as expected, there is no relative shift between (30) and (31).

The electron charge to mass quotient

The electron mass selected in (9) can be calculated with any of these two expressions

$$m_{e,18} = \frac{\mu_{0,e}\varepsilon_0 v_{1,e}h}{2\pi r_{e,e}} = \frac{2h}{\alpha_e^2 \lambda_{\infty,e}c} = 9.109\ 383\ \mathbf{7015} \times 10^{-31} \text{ (kg)} \quad (32)$$

Then, using the charge expression in (5), the electron charge to mass quotient can be calculated with these three expressions to give

$$e/m_{e,18} = -\frac{2\pi r_{e,e}\sqrt{2}}{\mu_{0,e}\sqrt{\varepsilon_0 v_{1,e}}h} = -\alpha_e^2 c \lambda_{\infty,e} \sqrt{\frac{\varepsilon_0 v_{1,e}}{2h}} = -1.758\,820\,010\,772\,16 \times 10^{11} \left(\frac{\text{C}}{\text{kg}}\right) \quad (33)$$

The NIST link [echtomp](#) gives the following numerical value

$$e/m_{e,22} = -1.758\,820\,008\,38(55) \times 10^{11} \text{ (C/kg)} \quad (34)$$

with a relative shift of $1 - (e/m_{e,22})/(e/m_{e,18}) = 1.361 \times 10^{-9}$

From last expression in (32), using (20) and (28) gives another fine-structure constant physical interpretation having also the wave-particle duality, this time, for two energy entities defined above

$$\alpha_e = \left(\frac{2hc/\lambda_{\infty,e}}{m_{e,18}c^2}\right)^{1/2} = \left(\frac{2E_{\text{Ry},e}}{E_{e,e}}\right)^{1/2} = 7.297\,352\,5643\,3301 \times 10^{-3} \quad (35)$$

The Bohr magneton

Now, let's consider the NIST Bohr magneton μ_B ($\text{J} \cdot \text{T}^{-1}$) expression [Bohr magneton](#) and use it as follows

$$\mu_{B,e} = \frac{e\hbar}{2m_{e,18}} = 9.274\,010\,078\,362\,16 \times 10^{-26} \text{ (J} \cdot \text{T}^{-1}) \quad (36)$$

while for the current NIST electron mass

$$\mu_B = \frac{e\hbar}{2m_{e,22}} = 9.274\,010\,0657(29) \times 10^{-26} \text{ (J} \cdot \text{T}^{-1} \text{ or A} \cdot \text{m}^2) \quad (37)$$

where $1 - \mu_{B,e}/\mu_B$ is 1.361×10^{-9} .

These other four equations for $\mu_{B,e}$ provide the same numeric value as in (36)

$$\mu_{B,e} = \frac{\alpha_e^2 \lambda_{\infty,e} e c}{8\pi} = \frac{r_{e,e} \lambda_{\infty,e} e c}{8\pi a_{0,e}} = \frac{r_{e,e} e c^2}{2v_{1,e}} = \frac{0.5 r_{e,e} e c}{\alpha_e} \text{ (J} \cdot \text{T}^{-1}) \quad (38)$$

From the last expression, the following Sommerfeld constant physical interpretation in terms of two magnetic moment entities is written as

$$\alpha_e = \frac{0.5 r_{e,e} e c}{\mu_{B,e}} = 7.297\,352\,5643\,3301 \times 10^{-3} \left(\frac{\text{A} \cdot \text{m}^2}{\text{A} \cdot \text{m}^2}\right) \quad (39)$$

where the numerator term is a kind of Bohr mini magneton.

More from the vacuum magnetic permeability constant. The a_0/c time.

Combining (8) and the third expression in (18), reproduced here as follows,

$$\mu_{0,e} = \frac{2\alpha_e h}{e^2 c} \left(\frac{\text{N}}{\text{A}^2}\right) \quad (40)$$

$$h = 2\pi r_{e,e} m_{e,18} c / \alpha_e \text{ (J} \cdot \text{s)} \quad (41)$$

and substituting $\alpha_e h$ from (41) in (40) produces this cute, all-electron terms, expression for the vacuum magnetic permeability yielding this numeric performance

$$\mu_{0,e} = 4\pi \frac{r_{e,e} m_{e,18}}{e^2} = 4\pi (0.999\,999\,999\,866\,774 \times 10^{-7}) \left(\frac{\text{m} \cdot \text{kg}}{\text{C}^2} \text{ or } \frac{\text{m} \cdot \text{kg}}{\text{s}^2 \cdot \text{A}^2} \right) \quad (42)$$

For the NIST numeric values, this becomes

$$\mu_0 = 4\pi \frac{r_{e,e} m_{e,20}}{e^2} = 4\pi (0.999\,999\,999\,881\,15 \times 10^{-7}) \left(\frac{\text{m} \cdot \text{kg}}{\text{C}^2} \text{ or } \frac{\text{m} \cdot \text{kg}}{\text{s}^2 \cdot \text{A}^2} \right) \quad (43)$$

for a relative numeric shift of $1 - \mu_{0,e}/\mu_0 = 1.438 \times 10^{-11}$ only. See (9), (10) and (17).

Finally, it can be derived from (41) another Sommerfeld constant physical interpretation with a particle-nature entity of unit of action being smaller than the reduced Planck quantum unit of action given by

$$\alpha_e = \frac{2\pi r_{e,e} m_{e,18} c}{h} = \frac{h_{m,2}}{h} \quad (44)$$

Then, equating (44) and (2) provides other exact expressions for the electron charge and the speed of light given by

$$e = c \left(4\pi \epsilon_0 r_{e,e} m_{e,18} \right)^{1/2} \quad (\text{C}) \quad (45)$$

and a length/time physical interpretation for c in terms of the H atom radius as follows

$$c = \frac{e}{\left(4\pi \epsilon_0 r_{e,e} m_{e,18} \right)^{1/2}} \left\{ \frac{a_{0,e}}{a_{0,e}} \right\} = \frac{a_{0,e}}{\left(r_{e,e} m_{e,18} / F_{C,e} \right)^{1/2}} = \frac{a_{0,e}}{\tau_0} \left(\frac{\text{m}}{\text{s}} \right) \quad (46)$$

where the time $\tau_0 = \left(r_{e,e} m_{e,18} / F_{C,e} \right)^{1/2}$ (s) was introduced.

The proton mass and charge radius

Let's now introduce the proton mass numerical value from COADATA 2018 and the current one on NIST link [pm22](#) as follows

$$m_{p,18} = 1.672\,621\,923\,69 \times 10^{-27} \text{ (kg)} \quad (47)$$

$$m_{p,22} = 1.672\,621\,925\,95 \times 10^{-27} \text{ (kg)} \quad (48)$$

where their relative numeric value shift is $1 - m_{p,18}/m_{p,22} = 1.351 \times 10^{-9}$.

The proton rms charge radius numeric value in COADATA 2014 [pcr14](#), 2018 [pcr18](#) and 2022 [pcr22](#) NIST link has been as follows

$$r_{p,14} = 8.751(61) \times 10^{-16} ; r_{p,18} = 8.414(19) \times 10^{-16} ; r_{p,22} = 8.4075(64) \times 10^{-16} \text{ (m)} \quad (49)$$

respectively. Using $r_{p,18}$ in the following h expression gives

$$h = 0.5\pi r_{p,18} m_{p,18} c \approx 6.627\,364\,736 \dots \times 10^{-34} \quad (50)$$

which is far from matching the exact h numerical value in (3).

Now, if we solve (50) for $r_{p,18}$ and calculate the “required” proton charge radius is

$$r_{pc} = \frac{2h}{\pi m_{p,18} c} = 8.412\ 356\ 413\ 422\ 37 \times 10^{-16} \text{ (m)} \quad (51)$$

The same numerical value is obtained with

$$r_{pc} = 4 \frac{r_{e,e} m_{e,18}}{\alpha_e m_{p,18}} = 4 \frac{\alpha_e a_{0,e}}{\mu_{18}} = \frac{\alpha_e^2 \lambda_{\infty,e}}{\pi \mu_{18}} = \frac{v_{1,e}^2 / c^2}{\pi R_{\infty,e} \mu_{18}} \text{ (m)} \quad (52)$$

where the following definition of the proton-electron mass ratio was introduced as follows

$$\mu_{18} = \frac{m_{p,18}}{m_{e,18}} = 1836.152\ 673\ 440\ 000\ 00 \approx^{\text{NIST}} 1836.152\ 673\ 426(32) . \quad (53)$$

The NIST current numerical value was downloaded from [p-emassr](#).

Additionally, from (18) and (22), the electron mass can be calculated and defined with these expressions

$$m_{e,18} = \frac{\alpha_e \hbar}{r_{e,e} c} = \frac{2R_{\infty,e} h}{\alpha_e^2 c} = 9.109\ 383\ 7015\ 0000 \times 10^{-31} \text{ (kg)} \quad (54)$$

which matches the numeric value in (9) and

$$m_{p,18} = m_{e,18} \mu_{18} = 1.672\ 621\ 9236\ 9000 \times 10^{-27} \text{ (kg)} \quad (55)$$

which matches (47).

The Boltzmann constant and the Stefan-Boltzmann constant

Finally, using the Boltzmann constant k_B (J/K) [Bc](#)

$$k_B = 1.380\ 6493 \times 10^{-23} \text{ (exact)} \quad \left(\frac{\text{J}}{\text{K}} \right) \quad (56)$$

which has units of energy divided by absolute temperature, the same as for entropy, and is widely used in thermodynamics and in semiconductor device theory, and the Stefan- Boltzmann constant σ ($\text{W} \cdot \text{m}^{-2} \cdot \text{K}^{-4}$) [SBc](#) defined as

$$\sigma_{SB} = \frac{\pi^2}{60} \frac{k_B^4}{\hbar^3 c^2} = 5.670\ 379\ 3476\ 2860 \times 10^{-8} \text{ (exact)} \quad \left(\frac{\text{W}}{\text{m}^2 \text{K}^4} \right) \quad (57)$$

used to describe the thermal emission of matter as a function of its absolute temperature. Here, for its numerical value, the 14-digit precision is stated as exact.

Solving (57) for k_B and using (51), (36), (22), (20), (17) and (12) to replace h and c factors, the following expression can be obtained

$$\alpha_e = \frac{(15/16/\pi^5) m_{e,18} v_{1,e} \lambda_{\infty,e} e^4 \varepsilon_0^{-2} \sigma}{k_B^4} = \frac{(15/\pi^3) m_{e,18} v_{1,e} \lambda_{\infty,e} a_0^4 F_{C,e}^2 \sigma}{k_B^4} \frac{(\text{J}^4 \cdot \text{K}^{-4})}{(\text{J}^4 \cdot \text{K}^{-4})} \quad (58)$$

whose calculations give an exact match to (1).

A few final expressions

a) The reduced Planck constant as the product of three geometrical means looks like this

$$\hbar = 0.5 \sqrt{r_{pc} a_{0,e}} \sqrt{m_{p,18} m_{e,18}} \sqrt{v_{1,e} c} \quad (\text{exact}) \quad \left(\frac{\text{J} \cdot \text{s}}{\text{rad}} \right) \quad (59)$$

b) The numerical value of the inverse of the extended Sommerfeld constant compared with the NIST [isc](#) gives

$$\frac{1}{\alpha_e} = 137.035\,999\,1789\,7000 \approx^{\text{NIST}} 137.035\,999\,177(21) \quad (60)$$

c) H_{21} (m) is the hydrogen atom 21 cm wavelength produced in the intergalactic and interstellar cold space - near 3 °K - which can be calculated by

$$H_{21} = \frac{8\sqrt{2}}{\pi} \frac{m_{e,18}}{m_{p,18}} \frac{a_{0,e}}{r_{pc}} \frac{c}{v_{1,e}} \frac{\lambda_{\infty,e}}{\alpha_e} = 0.211\,127\,497\,981\,252 \text{ (m)} \quad (61)$$

and, with these expressions too

$$H_{21} = 8\sqrt{2} \frac{a_{0,e}}{\alpha_e^4} = 8\sqrt{2} \frac{r_{e,e}}{\alpha_e^6} = 8\sqrt{2} \frac{a_{0,e}^3}{r_{e,e}^2} = \frac{8\sqrt{2}}{\pi} \frac{a_{0,e}^2 \lambda_{\infty,e}}{\mu_{18} r_{e,e} r_{pc}} = \frac{\sqrt{2}}{2} \frac{\mu_{18}^2 r_{pc}^2}{\alpha_e^6 a_{0,e}} = \frac{\sqrt{2}}{2} \frac{\mu_{18}^2 a_{0,e}^2 r_{pc}^2}{r_{e,e}^3} \quad (62)$$

Note that (62) gives expressions among two lengths with nearly 14 orders of magnitude apart.

From (61), the Sommerfeld constant can be expressed with five quotients entities, namely, with numbers, masses, radii, speeds and wavelengths as follows

$$\alpha_e = \frac{8\sqrt{2}}{\pi} \frac{m_{e,18}}{m_{p,18}} \frac{a_{0,e}}{r_{pc}} \frac{c}{v_{1,e}} \frac{\lambda_{\infty,e}}{H_{21}} \quad (63)$$

d) The proton to electron mass ratio can be represented in terms of other physical constants as follows

$$\begin{aligned} \mu_{18} &= \frac{16\pi a_{0,e}^2}{r_{pc} \lambda_{\infty,e}} = \frac{4r_{e,e}}{\alpha_e r_{pc}} = \frac{4\alpha_e a_{0,e}}{r_{pc}} = \frac{4\sqrt{r_{e,e} a_{0,e}}}{r_{pc}} = \frac{2\lambda_{C,e}}{\pi r_{pc}} = 4 \left(\frac{a_{0,e}}{r_{pc}} \right) \left(\frac{v_{1,e}}{c} \right) \\ &= \frac{2a_{0,e} e^2 / \epsilon_0 r_{pc} c}{h} = \left(\frac{8\sqrt{2}}{\pi} \right) \left(\frac{a_{0,e}}{r_{pc}} \right) \left(\frac{c}{v_{1,e}} \right) \left(\frac{\lambda_{\infty,e}}{H_{21}} \right) \left(\frac{h}{e^2 / 2\epsilon_0 c} \right) \end{aligned} \quad (64)$$

where (2) was used for the last two expressions derivation.

e) Finally, the speed of light in vacuum can be calculated with this expression having 11 physical constants

$$c = \left(\frac{7.5}{\pi^5} \frac{\sigma \mu_{B,e} m_{e,18} a_{0,e} e^5}{\alpha_e \epsilon_0^3 r_{e,e} r_{pc} m_{p,18} k_B^4} \right)^{1/2}. \quad (65)$$

Appendix A.

Table A1. Some hydrogen related lengths and their corresponding α -expression.

Length definition	Sym.	Numerical value	α - Expression
The H 21 cm spectral line in L band	$H_{21} =$	2.11127497981251E-01	$= (1/\alpha_e)^3 (2^{3/2}/\pi) \lambda_{\infty,e}$
	$H_{21} =$		$= (1/\alpha_e) (2^{3/2}/(\pi^2 \mu_{18} r_{pc})) (\lambda_{\infty,e})^2$
Rydberg wavelength	$\lambda_{\infty,e} =$	9.11267051824680E-08	$= \lambda_{\infty,e} = (1/\alpha_e) 4\pi a_{0,e}$
Bohr H radius	$a_{0,e} =$	5.29177211264680E-11	$= (\alpha_e/4\pi) \lambda_{\infty,e}$
Compton e-wavelength	$\lambda_{C,e} =$	2.42631023868309E-12	$= (\alpha_e^2/2) \lambda_{\infty,e} = h/m_{e,18}c$
Classical e-radius	$r_{e,e} =$	2.81794032429536E-15	$= (\alpha_e^3/4\pi) \lambda_{\infty,e}$
Compton p-wavelength	$\lambda_{C,p} =$	1.32140985538933E-15	$= (\alpha_e^2/2\mu_{18}) \lambda_{\infty,e} = h/m_{p,18}c$
Proton charge radius	$r_{pc} =$	8.41235641342237E-16	$= (\alpha_e^2/\pi\mu_{18}) \lambda_{\infty,e}$

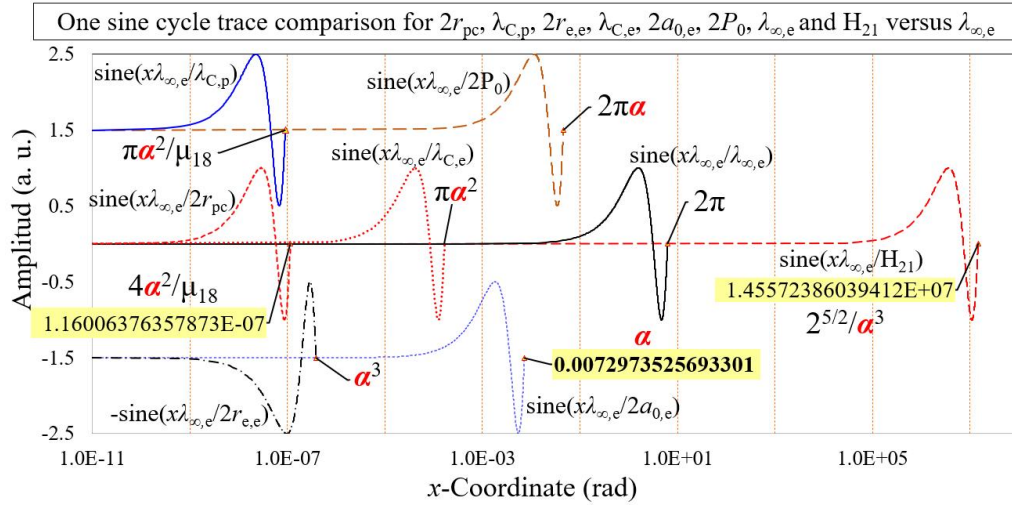


Fig. A1. A bit of Trigonometry with plenty of Physics. For the sake of clarity, four functions are displaced from the amplitude zero axis.