



SCIREA Journal of Physics

ISSN: 2706-8862

<http://www.scirea.org/journal/Physics>

October 28, 2026

Volume 11, Issue 5, October 2026

<https://doi.org/10.54647/physics140742>

The Role of Energy Cascade on the Uncertainty Principle: One Step Closer to Meet the Shy Graviton

Asghar Noormohammadi ^{1,*}

¹Department of Mechanical and Mechatronics Engineering, Univ. Waterloo, Waterloo, On, Canada

* Corresponding author: asghar13nasir@gmail.com

Abstract

In this research, we have made efforts to construct a model that formulates an equation for the motion of particles in quantum field theory (QFT). With this objective in mind, the Heisenberg uncertainty principle was examined for its relevance to the consolidation of all the presented fields. The result suggests a novel principle when the accumulated energy of all fields shows a decline as the frequency increases (energy cascade phenomena). Following this, we pinpoint that the canonical Heisenberg (Robertson) inequality, under the interaction of two electrons in the context of quantum electrodynamics (QED), leads to the enhanced Robertson-Schrodinger inequality that highlights the novel uncertainty principle with the cascade operator as the cornerstone of the quantum system. We posit that this may serve as a candidate for the graviton fields.

Keywords: Energy Cascade, Uncertainty Principle, Quantum Gravity, Quantum Electrodynamics

I. Introduction

A model that may bestow upon us a thorough description of the forces interacting within sub-atomic particles remains a focal aim for many studies. While the general theory of relativity successfully describes

large-scale motion, it has revealed certain weaknesses in its ability to account for gravitational effects at extremely small scales. Quantum mechanics has also demonstrated certain challenges when addressing the gravitational force. The gravitational field gives rise to a force that has yet to be fully addressed within the framework of quantum field theory (QFT). Figure (1) presents the Bronshtein cube, which illustrates various pathways to propose the quantum gravity derived from the general theory of relativity (relativity (c) and classical gravity (G)), alongside another perspective from quantum mechanics ($\frac{h}{2\pi}$) and relativity. In this study, the exploration of energy cascade phenomena when linked to the Heisenberg uncertainty principle has revealed a new principle.

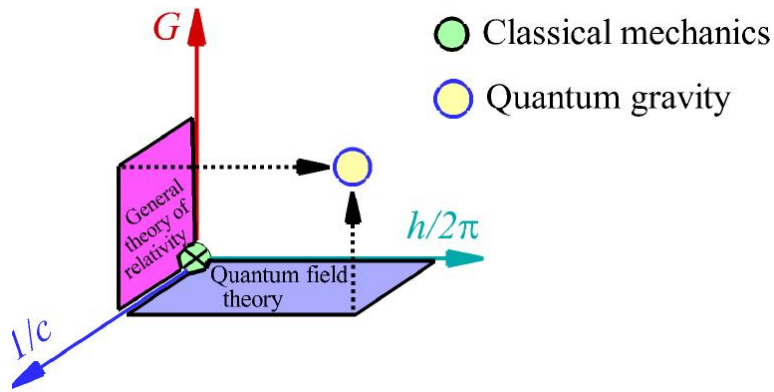


Figure 1: Schematic of Bronshtein cube.

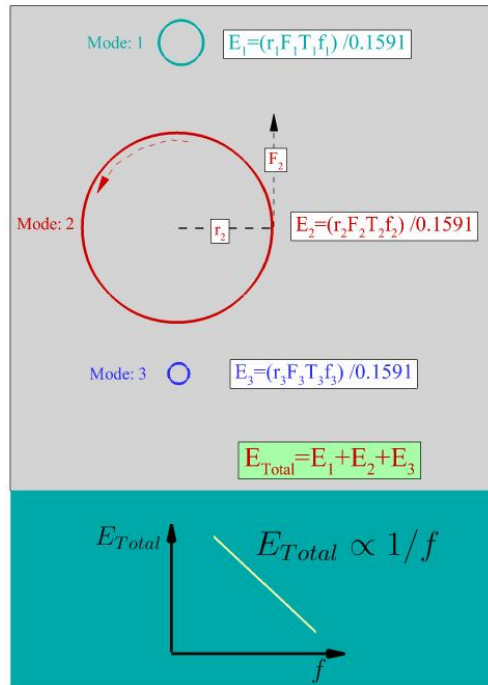


Figure 2: A representation of the total energy candidate (energy cascade) acquired by combining three modes.

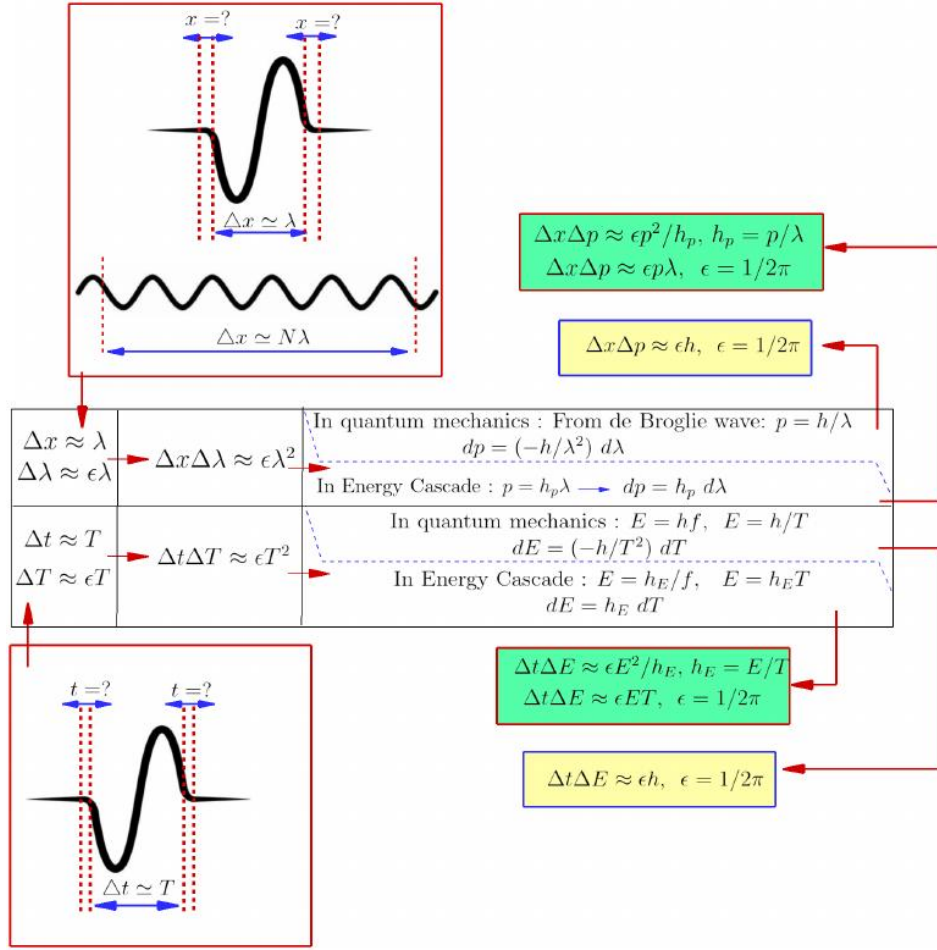


Figure 3: The nomenclature showing the uncertainty in momentum and energy under conditions of an energy cascade formation, along with its contrast to the Heisenberg uncertainty principle.

In the field of continuum-wave mechanics, it is observed that energy experiences a dynamic transfer among various scales present within a given system. This process of energy exchange frequently serves as a mechanism for transforming energy, facilitating the transition from larger scales, where energy is more readily transmitted, to smaller scales. This physical characteristic is recognized formally and is called the energy cascade [1, 2, 3, 4], a concept that was originally introduced by Richardson [13]. Furthermore, there are specific instances in which the phenomenon of backscatter energy is seen, illustrating the movement of energy as it transitions from smaller scales to larger scales [22]. Consequently, as this transition occurs within the system, the total energy manifests itself as if it is evolving into an energy cascade—a process characterized by a noticeable decrease in total energy levels that correlates with an increase in frequency, suggesting a relationship between energy distribution and frequency modulation. This intricate behavior of energy within wave mechanics highlights the essential principles governing energy dynamics and transformation across various scales. Understanding these interactions not only provides insight into the fundamental nature of energy but also informs broader applications within the realm of physics and engineering. In the fluid mechanics framework, the energy cascade is defined by non-linear interactions,

where the dynamics of energy at the smallest scales are largely influenced by viscosity, with dissipation determined as heat at the Kolmogorov microscale [5], thus enriching the internal energy. Furthermore, as the eddies reach their smallest possible size, there is an observable increase in their corresponding frequencies, accompanied by a decrease in energy levels. This results in the establishment of an isotropic condition

Through our study, we aim to illuminate the question of what the outcome is of integrating the cascade operator into the wave-dynamical nature of the quantum system, as it relates to the Heisenberg uncertainty principle.

The paper is organized as follows. In Section (II), we will review the basic concepts in energy cascade, thereby also illustrating the relation with the Heisenberg Uncertainty Principle. We then construct the computational aspects of the idea by utilizing the Dirac equation for the interaction of two electrons in Section (III). In this Section, we provide a new approach to the modified uncertainty principle in the quantum energy cascade frame. Finally, we discuss our findings in Section (IV).

II. Energy Cascade and its connection to Heisenberg uncertainty principle

Consider Fig. (2), depicting a sample candidate for the total energy E_{Total} . In this particular case, the total energy has been generated through the interaction of three modes and their respective fields. Each mode that encompasses a spatial size with a radius of r engages in rotation at the frequency f . When the lifetime of mode "2" is represented by the value of T_2 , the corresponding energy can be derived from $E_2 = \frac{r_2 F_2 T_2 f_2}{0.1591}$. F_2 embodies the concept of a living force. By integrating the energy associated with all three modes, one may observe the emergence of energy cascade phenomena. According to the energy cascade phenomena, the total energy diminishes as frequency increases. The energy cascade may find its expression within the framework of quantum field theory (QFT) solely if there is an inclusion of all fermions, force fields, and the Higgs boson field. This contrasts with the relationship observed between photon frequency and energy within a photon field or in any isolated field. In a system that encompasses only one fundamental field, augmenting frequency results in a corresponding rise in energy. Nevertheless, by integrating all essential fields of matter and forces, the energy-wave may evolve in a cascading manner. As such, it is essential to evaluate the total wave of any system that contains multiple fields and exhibits a cascading manner.

In turbulent eddies described by Kolmogorov's theory [5] and the concept of an energy cascade, larger eddies transfer not only more kinetic energy, but also momentum due to their larger scale. However, smaller eddies, despite having less individual momentum, are more numerous and could contribute remarkably to carrying momentum over a distance via turbulent mixing, which behaves like a continuous energy cascade from large to small scales. The relationship between a single eddy's momentum and its wavelength is not a direct proportionality, but rather a statistical one, where larger wavelength eddies are

associated with greater momentum content, and the overall momentum transfer is a result of the combined action of eddies across all scales in the turbulent flow [6]. The energy spectrum indicates that eddies with different wavelengths encompass different amounts of energy. The energy appears to be dominant within the spectrum at the smaller wavenumber, where the size of the eddies is comparatively larger [6]. This mechanism stands to gain from further elucidation via the following experimental thought. Envision yourself situated by the shore, next to the ocean. You shall adeptly vary the ocean's surface at an exceptionally high frequency. In parallel, a colossal being, far exceeding the size of a dinosaur, subtly influences the ocean's surface with low-frequency movements. The energy cascade will likely oversee the system's dynamics. You will be predominantly affected by the energy produced by the dinosaur. In addition to your involvement, should the authors and several others create fluctuations in the ocean, there may be instances where the dinosaur feels our presence more acutely. The dinosaur cannot sweat, but we certainly would experience that. Thus, a significant amount of matter and antimatter will influence the energy dynamics at a very small scale.

Although the energy cascade is mostly studied in classical fluid turbulence, it also occurs in quantum systems, such as in fluids of light, cold atoms, quantum gases, and quantum fluids [7, 8, 10]. In quantum systems, the energy cascade is distinguished by specific scaling laws (microscopic scale) [9] and is demonstrated through the dynamics of quantized vortex lines and the momentum distribution of particles. Furthermore, the notion of cascade processes is essential in understanding the relaxation dynamics of excited atomic systems. In multi-electron systems like the oxygen atom, energy redistribution after electronic excitation or inner-shell ionization is generally conveyed through distinct decay pathways, which include radiative relaxation as well as non-radiative processes such as the Auger effect and Coster--Kronig transitions [19, 20, 21]. Conventionally, these processes are regarded as sequential transitions among well-defined atomic states, under the framework of the time-dependent Schrödinger equation. In this context, these mechanisms engage the multi-electron Hamiltonian, which facilitates interaction-driven coupling among many-body eigenstates, resulting in a systematic redistribution of excitation energy across a hierarchy of quantum modes.

In the current work, inspired by the energy cascade concept, we propose a model of the Heisenberg uncertainty principle. Let us now examine the uncertainty principle in both its original formulation [11] and under the condition where total energy is in the form of an energy cascade. These two pathways are illustrated in Fig. (3). The uncertainty in space, denoted as Δx , and the uncertainty in wavelength, denoted as $\Delta \lambda$, can be represented by λ and $\epsilon \lambda$, respectively. This phenomenon occurs due to the formation of a wave with a single apex, resulting in a probability of particle presence on the scale of λ . In contrast, when the wave demonstrates N apexes, the likelihood of particle discovery in a broader space increase and becomes close to the magnitude of $N\lambda$. Similarly, when the wave takes the form of a single apex, the accuracy in determining λ decreases, leading to an uncertainty in wavelength ($\Delta \lambda$) that aligns with the scale of λ . However, if the wave assumes N apexes, the precision in determining λ improves, resulting in a reduction of uncertainty in wavelength ($\Delta \lambda$) to the order of $\epsilon \lambda$. In the realm

of quantum mechanics, ϵ is equivalent to $1/2\pi$. According to these connections between various spatial parameters, the following relationship can be deduced: $\Delta x \Delta \lambda = \epsilon \lambda^2$. Subsequently, it is necessary to establish a connection between λ and its corresponding momentum (p).

In the realm of quantum mechanics and in accordance with de Broglie waves, momentum is expressed as $p = h/\lambda$, where h is the Planck constant. On the flip side, in the framework of a turbulent energy cascade, de Broglie waves are not directly at play. This is because de Broglie waves are akin to the quantum-mechanical matter-wave nature of a single particle, such as an electron. Surveying the interaction among multiple particles, note that the system demonstrates a nonlinear energy cascade behavior, whereby energy is transferred reciprocally among various modes of waves in wave turbulence. As a result, the momentum is directly proportional to wavelength ($p = h_p \lambda$).

Following the first terminology in quantum mechanics and ignoring the minus sign [12], $\Delta x \Delta \lambda = \epsilon \lambda^2$ can be expressed as $\Delta x \Delta p \approx \epsilon h$. In contrast, by adopting the second approach in energy cascade, the following relationship can be derived: $\Delta x \Delta p \approx \epsilon p \lambda$. The diagram illustrated in Fig. (3) further shows that the energy-time pair of uncertainty has varied from its original formulation and adheres to the new principles as follows: $\Delta t \Delta E \approx \epsilon E T$, $\epsilon = 1/2\pi$. This reinforces the outcomes noted by [15]. According to Haug [haug2024compton], the Compton wavelength ($\lambda_c = h/mc$) can be characterized as the authentic matter wavelength. In an event where two prominent modes of the photon fields are in a dynamic interaction with an electron, the role of the Planck constant as a key factor in defining the Compton wavelength needs to be reconsidered. Thus, this signifies the point that the interaction of wavelengths from various fluctuations and fields specifically characterizes the Compton wavelength. The relationships and energy exchanges between the fluctuations in multiple fields lead to a phenomenology that mirrors the energy cascade addressed in the ongoing study.

III. Dirac Equation for Two Interacting Electrons

QED Lagrangian and Field Operators

Quantum Electrodynamics (QED) represents relativistic quantum field theory that elucidates the light-matter interaction, such as the interaction of charged spin $-\frac{1}{2}$ particles (electrons) with the electromagnetic field (photons) [16]. In the QED framework, the cascade concept refers to the situation in which electrons (e^-)/positrons (e^+) emit hard photons during motion with acceleration/deceleration by strong electromagnetic (EM) fields. In order to distinguish the QED cascade from the Unruh effect [14], notice that the Unruh effect describes how an accelerating observer detects a thermal bath of particles in vacuum (virtual particles), while a QED energy cascade is a physical procedure where high-energy photons/leptons create a chain reaction of secondary particles in a strong field or a medium (real particles).

In QED, the process by which electrons interact involves the exchange of photons. At the minimal order (tree-level), the interaction is conveyed by a virtual photon in Feynman diagrams. In Feynman diagrams, we contemplate two incoming electrons distinguished by momenta p_1 and p_2 , as well as the outgoing momenta

p_3 and p_4 . The Feynman diagrams comprise: (i) t -channel (characterized by photon exchange between legs) and (ii) u -channel (which entails the exchange of electrons in the final state). The core dynamical variables include the Dirac spinor field $\psi(x)$ for electrons and the gauge field $A_\mu(x)$ representing the characteristics of photons.

In natural units ($\hbar = c = 1$) this dynamical process is described by the QED Lagrangian density:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (1)$$

where m is the electron rest mass, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, and the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ couples electrons to the photon field vector A_μ with electric charge e . The gamma matrices γ^μ satisfy the Clifford algebra and implement relativistic spinor behavior.

In this framework, both ψ and A_μ are operator-valued fields acting on the QED Fock space, obeying canonical (anti-) commutation relations:

$$\{\psi_\alpha(x, t), \psi_\beta^\dagger(y, t)\} = \delta_{\alpha\beta}\delta^{(3)}(x - y), [A_\mu(x, t), A_\nu(y, t)] = 0, \quad (2)$$

where α, β denote spinor indices. From the QED Lagrangian density, it is possible to derive several key aspects, including: (i) the field equations via the Euler-Lagrange equations, (ii) perturbative calculation of Feynman rules, and (iii) conservation laws through the application of Noether's theorem. It should be emphasized that QED is recognized as a renormalizable theory, implying our ability to systematically resolve the infinities in loop calculations. In addition, the QED Lagrangian is established to be Lorentz invariant (consistent with special relativity), and Gauge invariant under local U(1) symmetry (preserving charge conservation). The gauge symmetry plays a pivotal role in facilitating the electron's engagement with the electromagnetic field, which inherently incorporates this interaction into the theoretical framework.

The process considered is:

$$e^-(p_1) + e^-(p_2) \rightarrow e^-(p_3) + e^-(p_4). \quad (3)$$

At tree level, there are two Feynman diagrams: one in the t -channel and one in the u -channel.

The matrix element $\mathcal{M}$ for a particle physics scattering process, likely an interaction involving four fermions, is given by,

$$\mathcal{M} = ie^2 \left[\frac{[\bar{u}(p_3)\gamma^\mu u(p_1)][\bar{u}(p_4)\gamma_\mu u(p_2)]}{(p_1 - p_3)^2} - \frac{[\bar{u}(p_4)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma_\mu u(p_2)]}{(p_1 - p_4)^2} \right], \quad (4)$$

here, $u(p_i)$ and γ^μ represent Dirac spinor for incoming/outgoing electron with momentum p_i and Dirac gamma metrics, respectively. Equation (4) represents the sum of two terms, each describing a different diagram in a Feynman diagram expansion. The first term is the t -channel diagram, and the second term is the u -channel diagram. The minus sign appears because electrons are identical fermions.

In the center-of-mass (CM) frame, the general formula for the differential cross section (probability of scattering) is,

$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} |\mathcal{M}|^2$, where $s = (p_1 + p_2)^2$ and $\mathcal{M}$ indicate the Mandelstam variable and the invariant amplitude, respectively [16].

In the ultra-relativistic regime, where the electron mass is negligible compared to the kinematic invariants ($s, |t|, |u| \gg m^2$), the squared matrix element for unpolarized Møller scattering takes the form [16]:

$$|\mathcal{M}|^2 = 2e^4 \left[\frac{s^2 + u^2}{t^2} + \frac{s^2 + t^2}{u^2} + \frac{2s^2}{tu} \right]. \quad (5)$$

So the differential cross section becomes:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \left[\frac{s^2 + u^2}{t^2} + \frac{s^2 + t^2}{u^2} + \frac{2s^2}{tu} \right], \quad (6)$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine-structure constant.

Before interaction, the state of the system consists of two free electron wave packets and no photons. The free wavefunction in spectral (momentum space) representation is given by

$$|\Psi_{free}\rangle = \int \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \phi_{in}(\vec{p}_1, \vec{p}_2) b^\dagger(\vec{p}_1) b^\dagger(\vec{p}_2) |0\rangle, \quad (7)$$

where $\phi_{in}(\vec{p}_1, \vec{p}_2)$ is the initial momentum-space wavefunction of the two electrons with the annihilation operator b . This function typically peaks around the incoming momenta and describes the spectral content of the unperturbed state.

As the electrons interact via photon exchange, the wavefunction becomes populated by virtual photon modes. The interacting state can be expressed in Fock space as

$$|\Psi_{int}\rangle = \sum_{n=1}^{\infty} \int \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \prod_{j=1}^n \frac{d^3 k_j}{(2\pi)^3} \phi^{(n)}(\vec{p}_1, \vec{p}_2; \vec{k}_1, \dots, \vec{k}_n) \times b^\dagger(\vec{p}_1) b^\dagger(\vec{p}_2) \prod_{j=1}^n a^\dagger(\vec{k}_j) |0\rangle, \quad (8)$$

where $\phi^{(n)}(\vec{p}_1, \vec{p}_2; \vec{k}_1, \dots, \vec{k}_n)$ represents the spectral amplitude for the two electrons exchanging n virtual photons with momenta $\vec{k}_j$, and the annihilation operator a . This term captures all orders of perturbative

photon exchange, enabling a cascade-like transfer of energy and momentum within the quantum field system. Equation (8) illustrates a refined Fock-space expansion that is restricted to the two-electron sector alongside an arbitrary number of photons. The processes related to virtual fermion–antifermion pair creation are not specifically included, and we posit that their effects are adequately integrated in the interaction coefficients $\phi^{(n)}$.

After the interaction, the photon field is no longer entangled with the electron pair, and the outgoing state can be described by a two-electron wavefunction in the asymptotic (free) limit:

$$|\Psi_{out}\rangle = \int \frac{d^3p_1'}{(2\pi)^3} \frac{d^3p_2'}{(2\pi)^3} \phi_{out}(\vec{p}_1', \vec{p}_2') b^\dagger(\vec{p}_1') b^\dagger(\vec{p}_2') |0\rangle, \quad (9)$$

where $\phi_{out}(\vec{p}_1', \vec{p}_2')$ is the final amplitude of the wavepacket in momentum space, incorporating all phase shifts and scattering effects accumulated during the interaction. This state represents the free asymptotic configuration of the electron system after the exchange of virtual photons has ceased.

Time Evolution of the Wavefunction and Spectral Energy Distribution

The standard framework for evaluating time-dependent approaches in QED is the interaction picture, which separates the time variation of the states and the interaction operators, providing us with a comprehensible description of how interactions change the state of the system over time.

The complete time-dependent evolution of the quantum state in QED for two interacting electrons can be expressed in three stages:

i. Before interaction ($t < t_0$), the particles are in "Free Evolution" and the free wave function in the spectral momentum space (Eq. 7), is given by

$$|\Psi_{free}(t < t_0)\rangle = \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \phi_{in}(\vec{p}_1, \vec{p}_2) \times e^{-i(E_{p_1} + E_{p_2})t} b^\dagger(\vec{p}_1) b^\dagger(\vec{p}_2) |0\rangle, \quad (10)$$

where ϕ_{in} is the initial spectral wavepacket and $E_{p_i} = \sqrt{\vec{p}_i^2 + m^2}$. In this stage, the spectral energy distribution of the free electrons ($\mathcal{E}_{in}$) is defined as

$$\mathcal{E}_{in} = \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} |\phi_{in}(\vec{p}_1, \vec{p}_2)|^2 \times (E_{p_1} + E_{p_2}). \quad (11)$$

ii. During interaction ($t_0 \leq t \leq t_1$), the stage is in the "Entangled Evolution", thus the Eq. (8) takes the form

$$|\Psi_{int}(t_0 \leq t \leq t_1)\rangle = \sum_{n=1}^{\infty} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \prod_{j=1}^n \frac{d^3k_j}{(2\pi)^3} f^{(n)} \times (\vec{p}_1, \vec{p}_2; \vec{k}_1, \dots, \vec{k}_n) e^{-i(E_{p_1} + E_{p_2} + \sum_j |\vec{k}_j|)t} b^\dagger(\vec{p}_1) b^\dagger(\vec{p}_2) \prod_{j=1}^n a^\dagger(\vec{k}_j) |0\rangle, \quad (12)$$

where $f^{(n)}$ encodes the entangled state of electrons, virtual fermion-antifermion pairs, and n virtual photons.

The spectral energy distribution during interaction (electron+virtual photon cascade) can be written as

$$\mathcal{E}_{int} = \sum_{n=0}^{\infty} \int \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \prod_{j=1}^n \frac{d^3 k_j}{(2\pi)^3} \times |f^{(n)}(\vec{p}_1, \vec{p}_2; \vec{k}_1, \dots, \vec{k}_n)|^2 \times (E_{p_1} + E_{p_2} + \sum_{j=1}^n |\vec{k}_j|). \quad (13)$$

In the earlier equations, $E_{p_i} = \sqrt{|\vec{p}_i|^2 + m^2}$ is the relativistic energy of the electron "i", and $|\vec{k}_j|$ is the energy of the virtual photon j^{th} (natural units $c = \hbar = 1$ used).

iii. After interaction ($t > t_1$), the two-electron wavefunction is in a "Scatter Free State", and Eq. (9) is refined as

$$|\Psi_{out}(t > t_1)\rangle = \int \frac{d^3 p_1'}{(2\pi)^3} \frac{d^3 p_2'}{(2\pi)^3} \phi_{out}(\vec{p}_1', \vec{p}_2') \times e^{-i(E_{p_1'} + E_{p_2'})t} b^\dagger(\vec{p}_1') b^\dagger(\vec{p}_2') |0\rangle, \quad (14)$$

with ϕ_{out} containing the scattering information and momentum redistribution. The spectral energy distribution following the state for outgoing electrons can be derived in a manner akin to Eq. (11).

Quantum Energy Cascade and a Modified Uncertainty Principle

Cascade Operator and Its Scaling Properties

In the quantum framework, a matrix element of an operator $\hat{O}$ between the initial and final states, $|i\rangle$ and $|f\rangle$ respectively, is defined as $\langle i | \hat{O} | f \rangle$, which gives the transition amplitudes by the operator. In scattering theory, based on Fermi's Golden Rule [17], the regarded operator is some interaction Hamiltonian or perturbation, and the square of the magnitude of the matrix element gives a probability of transition. Thus, if the spectral cascade is a kind of interacting mode mechanism, the matrix element would be defined in the same way as $\langle i | \hat{O}_{cascade} | f \rangle$, where $\hat{O}_{cascade}$ represents the cascade operator. Therefore, in this context, the energy spectral cascade refers to a procedure in which excitations or energy flow cascade across spectral modes, analogous to classical turbulence cascades but acting in mode space. Thus, in this case, the operator of the spectral cascade can be considered as coupling different modes, so the matrix elements are then the coupling strengths between two modes or states due to that cascade operator.

The Matrix element from Eq. (13) of spectral cascade is

$$|\mathcal{M}|^2 \sim \sum_{n=0}^{\infty} |f^{(n)}(\vec{p}_1, \vec{p}_2; \vec{k}_1, \dots, \vec{k}_n)|^2. \quad (15)$$

For later use, we define a positive-definite cascade operator

$$\mathcal{C}(t) \equiv \sum_{n=1}^{\infty} \int \prod_{j=1}^n \frac{d^3 k_j}{(2\pi)^3} |f^{(n)}(\{p\}; \{k_j\}; t)|^2 w(\{k_j\}), \quad (16)$$

where $w(\{k_j\})$ is a chosen weight function (for instance $w = 1$ counts occupation, while $w = \sum_j |k_j|$ counts energy). By construction $\mathcal{C}(t)$ measures the instantaneous spectral weight distributed over multi-photon channels and grows while the interaction populates many modes. In a cascade regime, $\mathcal{C}$ is dominated by high- n contributions and scales with the flux Π and with the characteristic interaction time τ_{int} as

$$\mathcal{C} \sim F(\alpha, e)(\Pi\tau_{int})^\gamma, \quad (17)$$

for some model-dependent function F and exponent $\gamma > 0$; the precise form requires either explicit resummation of diagrams or numerical evaluation.

Uncertainty Bounds Modified by Cascade for the Electron Subsystem

We now address how cascade-induced correlations between the electrons and the photon cloud modify uncertainty relations that refer only to electron observables. The canonical Heisenberg (Robertson) relation for two Hermitian operators A, B in a state ρ is

$$(\Delta A)_\rho (\Delta B)_\rho \geq \frac{1}{2} |Tr[\rho[A, B]]|. \quad (18)$$

Applied to the single-particle position x and momentum p , the commutator $[x, p] = i\hbar$ yields the usual bound. The Heisenberg–Robertson relation acts as a specific instance in the Robertson–Schrödinger inequality when the anticommutator (covariance) term is omitted. Crucially, Eq. (18) uses the full (reduced) state ρ for the degrees of freedom being measured. Consider the reduced density matrix of the two-electron subsystem obtained by tracing out photons

$$\rho_e(t) = Tr_\gamma[|\Psi(t)\rangle\langle\Psi(t)|]. \quad (19)$$

Even though the fundamental operator commutation relations for the electron operators remain $[x, p] = i\hbar$, the statistical right-hand side of Eq. (18) is evaluated with ρ_e . Correlations (and classical/quantum noise) induced by tracing out the photon cloud can enlarge the variances $(\Delta x)_{\rho_e}$ and $(\Delta p)_{\rho_e}$ and can also produce nonzero covariance terms entering the stronger Robertson–Schrödinger inequality

$$(\Delta x)^2 (\Delta p)^2 \geq \left(\frac{1}{2} |\langle[x, p]\rangle|\right)^2 + \left(\frac{1}{2} \langle\{\Delta x, \Delta p\}\rangle\right)^2. \quad (20)$$

Here, the anti-commutator term measures the symmetrized covariance between x and p . For an electron entangled with a photon cloud produced by a cascade process, the covariance term can become non-negligible and, in fact, may dominate the second term on the right-hand side. Denote the covariance by

$$Cov_{xp} \equiv \frac{1}{2} \langle\{\Delta x, \Delta p\}\rangle_{\rho_e}. \quad (21)$$

We write the Robertson–Schrödinger bound as

$$\Delta x \Delta p \geq \sqrt{\left(\frac{\hbar}{2}\right)^2 + Cov_{xp}^2}. \quad (22)$$

Equation (22) represents an extra modified term of Cov_{xp}^2 to the fundamental Planck's constant $\hbar$. In this regard, we may define an effective quantum of action [18]

$$\hbar_{eff}(t) \equiv 2\sqrt{\left(\frac{\hbar}{2}\right)^2 + Cov_{xp}^2} = \sqrt{\hbar^2 + 4Cov_{xp}^2}. \quad (23)$$

Hence, the operational lower bound on the product of uncertainties for the electron subsystem can be written compactly as

$$\Delta x \Delta p \geq \frac{1}{2} \hbar_{eff}(t). \quad (24)$$

This expression shows that even though the canonical commutator remains $i\hbar$ at the fundamental level, the effective uncertainty witnessed by an observer restricted to the electron degrees of freedom can be parametrized by an $\hbar$ -like quantity that depends on the entanglement-induced covariance. As such, the energy cascade can be viewed as a contributor to the discourse on quantum gravity. Hence, in this and the next sections, we seek to establish a link between the energy-cascade process and the entanglement-induced covariance seen in Eq. (22).

In quantum field theory, as formulated in Peskin and Schroeder [16], the fundamental dynamical quantity is the action functional

$$S[\phi] = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi), \quad (25)$$

where $S[\phi]$ is the action functional of the field configuration $\phi(x^\mu)$. Here $d^4x = dt d^3x$ denotes the spacetime volume element with coordinates $x^\mu = (t, x)$. The function $\mathcal{L}(\phi, \partial_\mu \phi)$ is the Lagrangian density depending on the field and its spacetime derivatives $\partial_\mu \phi$. $S[\phi]$ governs quantum amplitudes through the path integral

$$\mathcal{A} \sim \int \mathcal{D}\phi \exp\left(\frac{i}{\hbar} S[\phi]\right). \quad (26)$$

Here, $\hbar$ sets the fundamental scale controlling phase oscillations in configuration space and determines the effective quantum of action in phase space.

In canonical phase-space variables (x, p) , quantum fluctuations are constrained by the uncertainty relation

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \quad (27)$$

For quantum states with nontrivial correlations between position and momentum, the covariance

$$Cov_{xp} = \frac{1}{2} \langle \{x, p\} \rangle - \langle x \rangle \langle p \rangle \quad (28)$$

quantifies phase-space shear induced by interactions and dynamical mixing.

This motivates an effective phase-space uncertainty scale defined earlier in Eq. (23). This quantity can be interpreted as a state-dependent effective quantum of phase-space area, arising from correlation-induced deformation of the canonical (x, p) structure.

In interacting quantum field theories, nonlinear mode coupling generates correlations between conjugate variables, effectively modifying the quadratic fluctuations around classical paths,

$$S \approx S_{cl} + \frac{1}{2} \delta\phi \frac{\delta^2 S}{\delta\phi^2} \delta\phi, \quad (29)$$

which control Gaussian fluctuations in the path integral formulation. Here S_{cl} is the classical action evaluated at ϕ_{cl} , and $\delta\phi = \phi - \phi_{cl}$ is the field fluctuation. The operator $\frac{\delta^2 S}{\delta\phi^2}$ is the second functional derivative (Hessian), governing quadratic fluctuations in the path integral. Such correlations increase phase-space mixing, leading to an enlarged effective uncertainty scale characterized by $\hbar_{eff}$. In this interpretation, energy transfer and mode-coupling processes (energy cascade mechanisms) manifest as increased covariance between conjugate variables, thereby modifying the effective scale governing uncertainty relations, while the fundamental constant $\hbar$ remains unchanged.

In order to find the relation between the covariance term to the cascade operator, we should mention that the covariance Cov_{xp} is a state functional whose magnitude is controlled by how strongly the electron position and momentum become correlated through interactions with intermediate photons. We can therefore posit the following relation at the level of scaling and dimensional analysis

$$Cov_{xp}(t) \simeq G_{cascade} \mathcal{W}(\mathcal{C}(t)), \quad (30)$$

where $\mathcal{C}(t)$ is the cascade operator defined in Eq. (16), $\mathcal{W}$ is a model-dependent positive function that maps spectral weight into an action-like quantity, and $G_{cascade}$ is a dimensionful coefficient that sets the coupling scale between cascade weight and electron phase-space covariance. Inserting Eq. (30) into Eq. (23) gives

$$\hbar_{eff}(t) = \sqrt{\hbar^2 + 4(G'_{cascade})^2 \mathcal{W}^2(\mathcal{C}(t))}. \quad (31)$$

For small cascade-induced covariance ($G'_{cascade} \mathcal{W} \ll \hbar$) one may expand

$$\hbar_{eff}(t) \approx \hbar + \frac{2(G'_{cascade})^2}{\hbar} \mathcal{W}^2(\mathcal{C}(t)) + \mathcal{O}(\mathcal{C}^4). \quad (32)$$

Identifying the leading cascade correction with a single parameter $G_{cascade}$, we arrive at the phenomenological relation used earlier in the paper (Eqs. (22) and (30):

$$\hbar_{eff}(t) \simeq \hbar + G_{cascade} \mathcal{C}(t). \quad (33)$$

The derivation of Eq. (33) from Eq. (32) is contingent upon the model-dependent function $\mathcal{W}(\mathcal{C}(t))$ being zero at the zeroth order in powers of $\mathcal{C}(t)$ and non-zero at the first order. Furthermore, it is vital that $\mathcal{W}$ maintains its role as an analytical function to avert the appearance of negative powers of $\mathcal{C}(t)$. This identification encapsulates the main theoretical claim: the multi-scale population of intermediate virtual photon modes — the quantum energy cascade — produces nontrivial electron-photon correlations which in turn increase the effective minimal phase-space cell size for observables restricted to the electron subsystem. The cascade, therefore, contributes an

$\hbar$ -like term that can be interpreted as the action-scale of an emergent collective ("graviton-like") degree of freedom.

Cascade Operator from the Dirac wave (field) solution

We derive Eq. (16) starting from the QED interaction in the Dirac-field formalism and computing the leading-order cascade-induced contribution to the reduced electron covariance. In this regard, the Dirac field satisfies

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = e\gamma^\mu \psi(x)A_\mu(x), \quad (34)$$

and the interaction-picture Hamiltonian density is

$$\mathcal{H}_I(x) = -e\bar{\psi}(x)\gamma^\mu \psi(x)A_\mu(x) \equiv -j^\mu(x)A_\mu(x), \quad (35)$$

where $j^\mu(x) = e\bar{\psi}(x)\gamma^\mu \psi(x)$ is the electron current operator. Now, to derive the Dyson expansion and second-order contribution, let's use the time-evolution (Dyson) operator as

$$S = T \exp \left(-i \int d^4x \mathcal{H}_I(x) \right), \quad (36)$$

where T indicates the time-ordered product of operators. Acting on an initial two-electron state with no photons $|\Psi_0\rangle$ and expanding to second order in e gives the leading entangling term

$$|\Psi(t)\rangle \simeq |\Psi_0\rangle - i \int d^4x \mathcal{H}_I(x) |\Psi_0\rangle - \frac{1}{2} \int d^4x d^4y T[\mathcal{H}_I(x)\mathcal{H}_I(y)] |\Psi_0\rangle + \mathcal{O}(e^3). \quad (37)$$

The second-order term creates virtual-photon excitations and is the lowest-order source of the multimode photon cloud relevant for the cascade. The complete density matrix is $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$. Tracing out photons yields the reduced electron density matrix

$$\rho_e(t) = \text{Tr}_\gamma \rho(t). \quad (38)$$

Retaining terms up to e^2 one finds (schematically)

$$\rho_e(t) \simeq \rho_e^{(0)} + \rho_e^{(2)} + \mathcal{O}(e^3), \rho_e^{(0)} = |\Psi_0\rangle\langle\Psi_0|_e, \rho_e^{(2)} = \int d^4x d^4y \mathcal{K}^{\mu\nu}(x, y) J_\mu(x) \rho_e^{(0)} J_\nu(y)^\dagger, \quad (39)$$

where $J_\mu(x)$ denotes the electron-current operator acting on the electron Hilbert space (implicit spinor indices). The kernel $\mathcal{K}^{\mu\nu}(x, y)$ is obtained from the photon-field contraction (propagator):

$$\mathcal{K}^{\mu\nu}(x, y) = \frac{1}{2} \langle 0_\gamma | T(A^\mu(x)A^\nu(y)) | 0_\gamma \rangle = \frac{1}{2} D^{\mu\nu}(x - y). \quad (40)$$

The combinatorial $1/2$ depends on ordering conventions; the key structure is the insertion of the photon propagator coupling two electron-current insertions. In this step, for determining the electron observable covariance to $\mathcal{O}(e^2)$, let's define the symmetrized covariance for single-electron observables (we will focus on the center-of-mass / one-particle reduced operators or an appropriate single-electron marginal) given by

$$Cov_{xp}(t) \equiv \frac{1}{2} Tr_e[\rho_e(t)\{\Delta x, \Delta p\}], \quad (41)$$

where, $\Delta x \equiv x - Tr_e[\rho_e x]$, and $\Delta p \equiv p - Tr_e[\rho_e p]$. Expand Eq. (41) to second order using Eq. (39):

$$Cov_{xp}(t) \simeq Cov_{xp}^{(0)} + Cov_{xp}^{(2)} + \dots \quad (42)$$

For a prepared incoming product wavepacket ($\rho_e^{(0)}$) one can arrange $Cov_{xp}^{(0)}$ to be small (e.g. minimum-uncertainty packet). The leading cascade-induced contribution is

$$Cov_{xp}^{(2)} = \frac{1}{2} Tr_e[\rho_e^{(2)}\{x, p\}] - Tr_e[\rho_e^{(2)}x] Tr_e[\rho_e^{(0)}p] - Tr_e[\rho_e^{(0)}x] Tr_e[\rho_e^{(2)}p]. \quad (43)$$

By substituting the representation of Eq. (39) for $\rho_e^{(2)}$; and after moving traces inside and using cyclicity, one may discern the ensuing details

$$Cov_{xp}^{(2)} = \int d^4x d^4y \mathcal{K}^{\mu\nu}(x, y) \mathcal{M}_{\mu\nu}(x, y), \quad (44)$$

where the matrix element functional $\mathcal{M}_{\mu\nu}(x, y)$ is

$$\mathcal{M}_{\mu\nu}(x, y) \equiv \frac{1}{2} Tr_e[J_\mu(x) \rho_e^{(0)} J_\nu(y)^\dagger \{x, p\}] - \dots \quad (45)$$

(the dots denote the subtraction pieces that remove shifts of single expectation values; they have the same structure and are kept implicitly).

Nevertheless, we express the current insertions as matrix elements between Dirac spinor wave packets, characterized by the incoming two-electron wavefunction $\phi_{in}(p_1, p_2)$, such that $\mathcal{M}_{\mu\nu}(x, y)$ becomes an integral over electron momenta with factors involving $\bar{u}(p)\gamma_\mu u(p')$ and plane-wave phases. Inserting into Eq. (44) and performing the spacetime integrals yields a momentum-space expression of the schematic form

$$Cov_{xp}^{(2)} = e^2 \int \frac{d^4k}{(2\pi)^4} \tilde{D}^{\mu\nu}(k) \mathcal{S}_{\mu\nu}(k; \{\phi_{in}\}), \quad (46)$$

where $\mathcal{S}_{\mu\nu}(k; \{\phi_{in}\})$ encodes the overlap of electron current matrix elements, factors of x, p from the observable insertions, and the incoming wavepacket spectral shapes. Crucially, $\mathcal{S}$ is nonzero for a continuum of photon momenta k and therefore the integral collects contributions across modes. Consider that in the above result, the momentum-space Fourier transformation of the photon propagator is utilized.

In the following, compare Eq. (46) with the cascade operator defined earlier (Eq. (16)).

At leading (second) order, the dominant multi-photon weight reduces to the one-photon sector amplitude squared $|f^{(1)}(k)|^2$, and the kernel $\mathcal{S}_{\mu\nu}(k)$ in Eq. (46) is directly proportional to $|f^{(1)}(k)|^2$ times factors that encode how position and momentum couple to that mode. Thus, schematically,

$$Cov_{xp}^{(2)} \propto e^2 \int d^3k \omega_k |f^{(1)}(k)|^2 \mathcal{F}(k; \{\phi_{in}\}) \sim \tilde{G} \mathcal{C}_{(1)}(t), \quad (47)$$

where $\mathcal{C}_{(1)}$ denotes the one-photon (lowest-order) contribution to the cascade operator and $\mathcal{F}$ is a dimensionless mapping from spectral weight to an action-like covariance. Repeating the argument to

include higher orders (multi-photon terms $f^{(n)}$) shows that Cov_{xp} receives cumulative contributions proportional to the full $\mathcal{C}(t)$. We therefore posit the scaling relation (made explicit earlier)

$$Cov_{xp}(t) \simeq G'_{cascade} \mathcal{W}(\mathcal{C}(t)), \quad (48)$$

with $G'_{cascade}$ (a dimensionful coefficient absorbing coupling constants and phase-space factors).

Inserting the above covariance into the Robertson-Schrodinger inequality (Eq. (22))

and defining $\hbar_{eff}(t)$ as represented in Eq. (23), one recovers the operational bound

$$\Delta x \Delta p \gtrsim \frac{1}{2} \hbar_{eff}(t), \quad (49)$$

which is the same as Eq. (24), and after substituting $Cov_{xp} \propto \mathcal{C}(t)$ and expanding for small cascade corrections, we have

$$\hbar_{eff}(t) \simeq \hbar + G_{cascade} \mathcal{C}(t) + \mathcal{O}(\mathcal{C}^2). \quad (50)$$

The derivation above is perturbative and schematic: the explicit integrand $\mathcal{S}_{\mu\nu}(k)$ in Eq. (46) is obtained by inserting explicit two-electron Dirac spinor wavepackets (plane-wave or localized Gaussian packets) and carrying out the spinor traces. Doing so gives a closed integral expression that can be evaluated numerically for chosen wavepackets and kinematics.

It is worth mentioning that the cascade operator $\mathcal{C}(t)$ is nothing but the spectral weight of multi-photon amplitudes $|f^{(n)}|^2$. At $\mathcal{O}(e^2)$, the correspondence between $Cov_{xp}^{(2)}$ and $\mathcal{C}_{(1)}$ is direct; higher orders extend the proportionality and can build a flux-dominated cascade if many modes are populated.

IV. Discussion

In this study, we have developed a novel approach to the Heisenberg uncertainty principle in the energy cascade frame. The two-electron QED scattering problem, when examined beyond lowest order and with attention to the full multi-photon content of intermediate states, naturally gives rise to a broad spectral occupation of virtual photon modes. If the redistribution of spectral energy takes place in a scale-local, flux-dominated way, it is appropriate to call the process a quantum energy cascade. When electrons are treated as a subsystem (photons traced out), the cascade-generated electron--photon correlations produce covariance contributions to uncertainty bounds. These covariance terms augment the operational lower bound on $\Delta x \Delta p$, and can be compactly parametrized by an $\hbar_{eff}(t)$ that contains the usual Planck constant plus a positive contribution proportional to a cascade operator $\mathcal{C}(t)$. This provides a concrete pathway from a QED-based cascade picture to an effective modification of the uncertainty relation, and motivates the proposal that the cascade contribution may be interpreted phenomenologically as an emergent graviton-like action scale.

It is important to highlight that the interaction does not alter the canonical commutation relations of the fundamental field operators: $[x, p] = i\hbar$ remains true. The effect described in this context concerns the reduced uncertainty bound applicable to a subsystem due to entanglement and decoherence.

On the other hand, the identification of the cascade correction with a graviton-like degree of freedom is heuristic and phenomenological. A true identification would require demonstrating that the emergent collective mode satisfies the characteristics of a massless spin-2 field and couples universally to energy-momentum; the present derivation only argues for an emergent action scale associated with the cascade.

Moreover, quantitative evaluation of $\mathcal{C}(t)$ and Cov_{xp} requires either (i) explicit perturbative calculation of the multi-photon amplitudes $f^{(n)}$, including resummation of classes of diagrams that dominate in the cascade regime, or (ii) non-perturbative numerical simulation of a simplified (e.g., scalar) model to extract scaling. The formal expressions above provide a clear set of targets for such calculations.

Moreover, the cascade interpretation in the macroscopic regime can be discussed when the curvature argument is ignored or weakened. If the mass-energy distribution is homogeneous and static, then all gradients vanish, which means, $\nabla_\mu T^{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$, and yielding the flat Minkowski metric ($g_{\mu\nu} = \eta_{\mu\nu}$). Gravity, therefore, disappears in the limit of perfect energy uniformity. However, when matter and radiation exhibit a nonuniform flow, a "mass-energy cascade", the spatial gradients of mass-energy density ρ , flux density j_i , and finally the stress tensor σ_{ij} become nonzero. This introduces curvature via the Einstein equation of relativity.

For instance, defining the macroscopic cascade current

$$\mathcal{J}_M^\mu = -D_M \nabla_\nu T^{\mu\nu}, \quad (51)$$

with D_M an effective transport coefficient, the divergence of $\mathcal{J}_M^\mu$ acts as a source of curvature:

$$R \sim \frac{8\pi G}{c^4} \nabla_\mu \mathcal{J}_M^\mu. \quad (52)$$

Thus, curvature, and hence gravity, can be viewed as the geometric imprint of an ongoing cascade of mass and energy across spacetime. When this cascade vanishes ($\mathcal{J}_M^\mu = 0$), spacetime flattens, and gravity disappears.

In the quantum analogy of energy cascade and Graviton emergence, we should mention that, at the quantum scale, the energy flux between interacting fields (e.g., two electrons and a photon cloud) forms a "quantum energy cascade", analogous to $\mathcal{J}_M^\mu$ but defined in Hilbert space. The modified uncertainty relation was derived earlier, in $\Delta x \Delta p \gtrsim \frac{1}{2}(\hbar + G_{\text{cascade}} \mathcal{C}(t))$, introduces an effective curvature term $G_{\text{cascade}} \mathcal{C}(t)$ that parallels the macroscopic curvature sourced by $\nabla_\mu \mathcal{J}_M^\mu$ in General Relativity. Therefore, the "graviton" may be interpreted as the quantized excitation of the energy-cascade field, the microscopic analogue of spacetime curvature produced by cascading mass-energy. When all quantum modes equilibrate ($\mathcal{C}(t) \rightarrow 0$), the curvature term vanishes, returning the system to flat quantum space. Hence, this process contributes to the shift from the standard Heisenberg inequalities to the Heisenberg uncertainty limit $[x, p] = i\hbar$.

Finally, our research indicates that exploring the Heisenberg uncertainty principle in the context of energy cascade constructs a new representation of the uncertainty principle. It is noted that when the energy

cascade is evaluated, the right-hand side of uncertainty no longer indicates reliance on the Planck constant. The right-hand side appears to exhibit a degree of sensitivity to the newly introduced parameters related to momentum and wavelength in the first pair, as well as energy and time in the second pair of uncertainty. This new principle can undergo further investigation.

Data Availability Statements

Currently, we do not have any particular data to present. Nevertheless, we are open to addressing concerns or sharing information about the energy cascade pattern formation within the field, which contains multiple modes, if you'd like.

References

- [1] D. Park and A. Lozano-Duran, "The coherent structure of the energy cascade in isotropic turbulence," *Scientific Reports*, 15,1 2025.
- [2] S. Chen, R. E. Ecke, G. L. Eyink, M. Rivera, M. Wan, and Z. Xiao, "Physical Mechanism of the Two-Dimensional Inverse Energy Cascade," *Phys. Rev. Lett.* 96, 084502 (2006).
- [3] V. E. Zakharov, P. Guyenne, A. N. Pushkarev, and F. Dias, "One-dimensional wave turbulence," *Physics Reports*, 398, 1--65, 2004.
- [4] P. Tabeling, "Two-dimensional turbulence: a physicist approach," *Physics Reports*, 362, 1--62 2002.
- [5] N. D. Katopodes, "Chapter 8 - Turbulent Flow," *Free-Surface Flow*, N. D. Katopodes, Ed. Butterworth-Heinemann, 566--650 2019.
- [6] M. Hirota, S. Izawa, and Y. Fukunishi, "Response of turbulent energy spectrum and flow structures when vortical motion of a certain scale is suppressed by artificial forcing," *Physical Review Fluids*, 9, 084602, 2024.
- [7] R. Numasato, M. Tsubota, and V. S. L'vov, "Direct energy cascade in two-dimensional compressible quantum turbulence," <https://journals.aps.org/PhysRevA/abstract/10.1103/PhysRevA.81.063630> (2010).
- [8] S. K. Jha, M. K. Verma, S. I. Mistakidis, and P. K. Mishra, "Energy spectra and fluxes of two-dimensional turbulent quantum droplets," *Phys. Rev. Fluids* **10**, 064618 (2025).

- [9] A. Sivakumar, P. K. Mishra, A. A. Hujeriat, and P. Muruganandam, "Energy spectra and fluxes of turbulent rotating Bose--Einstein condensates in two dimensions," *Physics of Fluids*, 36(2) 2024.
- [10] X. Zhang, T. Zhu, H. Du, H.-G. Luo, J. van den Brink, and R. Ray, "Extended high-harmonic spectra through a cascade resonance in confined quantum systems," *Phys. Rev. Res.* 4, 033026 (2022).
- [11] W. Heisenberg, "Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik," *Zeitschrift für Physik* 43, 172--198 1927.
- [12] K. S. Krane, "Modern physics" John Wiley & Sons (2019).
- [13] L. F. Richardson, "Weather prediction by numerical process," University Press, 1922
- [14] {Unruh}, W. G. "Notes on black-hole evaporation," *Phys. Rev. D* 14, 870--892 (1976).
- [15] Espen Gaarder Haug, "The Compton Wavelength Is the True Matter Wavelength, Linked to the Photon Wavelength, While the de Broglie Wavelength is Simply a Mathematical Derivative, Understanding this Leads to Unification of Gravity and New Quantum Mechanics". *Qeios* (2024).
- [16] M. E. Peskin, D. V. Schroeder, "An Introduction to Quantum Field Theory," Westview Press, (1995).
- [17] E. Fermi, "Nuclear Physics: A Course Given by Enrico Fermi at the University of Chicago," University of Chicago Press, (1950).
- [18] J. M. Cline, "The Quantum Effective Action," in *Advanced Concepts in Quantum Field Theory: With Exercises*, Cham: Springer International Publishing, 2020.
- [19] G. Herzberg, "Atomic Spectra and Atomic Structure", Dover Publications (1944).
- [20] R. D. Cowan, *The Theory of Atomic Structure and Spectra*, University of California Press (1981).
- [21] M. Mehlhorn et al., "Auger- and Coster--Kronig Processes", Springer (1978).
- [22] A. Noormohammadi, "A priori testing of large eddy simulation in open channel turbulent flows," *Phys. Fluids* 38(3), (2026).