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The proton charge radius and the proton to electron mass ratio equations

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Abstract

New expressions in terms of the proton charge radius are derived for the Sommerfeld or fine-structure constant, the proton to electron mass ratio and the elementary particles Wirkungsquanta. Through the derivation of these expressions, it was found that a small correction factor was needed to be applied on the currently accepted proton charge radius numeric value so that the calculated α values were matched with good relative standard uncertainties. Quasi-exact matches to the h-value were used to confirm the numerical value of the used proposed proton charge radius.

Introduction.

Sommerfeld, in his first physical interpretation of the fine-structure constant, used the Bohr model of H to define α as the ratio of two speeds, namely, the first orbit electron speed, v_1 (m/s), and the speed of light in vacuum, c (m/s), as follows

$$\alpha \equiv \frac{v_1}{c} = \frac{e^2}{4\pi\varepsilon_0\hbar}\frac{1}{c} = 0.007\ \ 297\ \ 352\ \ 5693 \quad (\text{rsu } 1.5 \text{ x } 10^{-10}) \tag{1}$$

where e (C) is the electron charge, ε_0 (F/m) is the vacuum dielectric permittivity, c (m/s) is the speed of light in vacuum and $\hbar$ (Js) is the Planck constant h divided by 2π; rsu stands for relative standard uncertainty. The given numeric value is as internationally recommended by the 2018 CODATA [1]. At present, there are over 10 definitions for α [2]. None of which incorporates the proton size nor the proton to electron mass ratio, μ, whose numeric value is given by [1] µ NIST

$$\mu \equiv \frac{m_p}{m_e} = 1\ 836.152\ \ 673\ \ 43 \ \ (\text{rsu } 6.0 \text{ x } 10^{-11}) \tag{2}$$

where, m_p, the proton rest mass, has the value [1]

$$m_p = 1.672\ 621\ 923\ 69 \text{ x } 10^{-27} \text{ kg} \ \ (\text{rsu } 3.1 \text{ x} 10^{-10}) \tag{3}$$

and, m_e, the electron rest mass, is [1]

$$m_e = 9.109\ 383\ 7015 \text{ x } 10^{-31} \text{ kg} \ \ (\text{rsu } 3.0 \text{ x} 10^{-10}), \tag{4}$$

From just numerology to serious theoretical and experimental efforts have been conducted aiming to determine μ with high precision [3, 4, 5, 6, 7]. However, a μ calculating expression has not been disclosed.

The proton charge radius, r_{pc} (m), also lacks a calculating equation compounded with a very complex experimental determination of its value which gives rise to a high uncertainty level; this situation has been referred to as the proton radius puzzle [8, 9, 10, 11]. Derivation of an equation for r_{pc} will provide a useful approach to this issue.

Expressions for α and μ in terms of the proton charge radius.

For the forthcoming derivations, the following definition of α, given in [1] re NIST , will be used

$$\alpha^2 \equiv \frac{\text{classical electron radius}}{\text{Bohr radius for H atom}} = \frac{r_e}{a_0} \tag{5}$$

where

$$r_e = 2.817\ 940\ 3262 \text{ x } 10^{-15} \text{ m} \quad (\text{rsu } 4.5 \text{ x } 10^{-10}) \tag{6}$$

and [1] Bohr r NIST

$$a_0 = 5.291\ 772\ 109\ 03 \text{ x } 10^{-11} \text{ m} \quad (\text{rsu } 1.5 \text{ x } 10^{-10}) . \tag{7}$$

Let's write (5) as follows

$$\alpha\alpha = \left(\frac{r_e}{1}\right)\left(\frac{1}{a_0}\right) = (2.8179 \text{ x } 10^{-15})(1.8897 \text{ x } 10^{10}) \tag{8}$$

and aim at making the two terms in parenthesis on the right to coincide in the value of α by multiplying and dividing by selected parameters. Their starting ratio is about 10^{-25}.

Firstly, using the proton charge radius, r_p, whose NIST rms value is rp NIST

$$r_p = 8.414 \text{ x } 10^{-16} \text{ m} \quad (\text{rsu } 2.2 \text{ x } 10^{-3}), \tag{9}$$

to multiply and divide (8) gives

$$\alpha\alpha = \left(\frac{r_e}{r_p}\right)\left(\frac{r_p}{a_0}\right) = (3.3491)(1.590 \text{ x } 10^{-5}) \tag{10}$$

That was a huge swing, the terms ratio dropped 31 orders of magnitude and is now about 10^5.

Next, we use μ to multiply and divide (10) so that this expression turns into

$$\alpha\alpha = \left(\frac{r_e}{r_p\mu}\right)\left(\frac{r_p\mu}{a_0}\right) = (0.00182398\ 777)(0.029\ 1951132360\ 308) \tag{11}$$

Now, the ratio between the two factors is approximately 1/16. This calls for introducing a factor of 4 as follows

$$\alpha\alpha = \left(\frac{4r_e}{r_p\mu}\right)\left(\frac{r_p\mu}{4a_0}\right) = (0.0072959271\,081)(0.0072987783\,09) \tag{12}$$

The new ratio of the factors is

$$\kappa \equiv \frac{0.0072959271\,081}{0.0072987783\,09} = 0.9996093591\,56918 \tag{13}$$

Then, the required factor in (12) ought to be about

$$\kappa^{1/2} = \sqrt{\frac{0.0072959271\,081}{0.0072987783\,09}} = 0.9998066604\,99699 \tag{14}$$

With this, (12) becomes

$$\alpha\alpha = \left(\frac{4r_e}{(r_p\kappa^{1/2})\mu}\right)\left(\frac{(r_p\kappa^{1/2})\mu}{4a_0}\right) = \left\{\frac{4r_e}{r_{pc}\mu}\right\}\left\{\frac{r_{pc}\mu}{4a_0}\right\} = (\alpha)(\alpha) \tag{15}$$

Where the "expected" proton charge radius, r_{pc}, was defined and its numerical value, using (6) and (14), is written as

$$r_{pc} \equiv r_p\kappa^{1/2} = 8.412\ \ 356\ \ 413\ \ 42809 \text{ x } 10^{-16} \text{ m} \tag{16}$$

where the value in (14) was actually fine-tuned aimed at reaching the same uncertainty level for both bracketed factors in (15) so that the currently accepted proton radius given in (9) was multiplied by the factor 0.999804660497748.

Expressions in (15) provide a couple of new, in author's opinion, physical interpretations for Sommerfeld's constant determined by the ratio of two ratios, one of them is the proton to electron mass ratio, μ, while the other one is the ratio of two lengths with both expressions involving r_{pc}. The parameter dimension nature of the ratio in the numerator and the parameter dimension nature of the ratio in the denominator is interchanged in these definitions. However, after rearranging factors, this changes into two ratios of angular moments with both particles at same speed. Their corresponding numeric value are given by

$$\alpha = 4\frac{r_e / r_{pc}}{m_p / m_e} = 4\frac{r_e m_e}{r_{pc} m_p} = 4\frac{\rho}{\mu} = 0.007\ \ 297\ \ 352\ \ 5693\ \ \ \underline{142}\ \ (\text{rsu } 1.95 \text{ x } 10^{-12} \text{ vs } (1)) \tag{17}$$

and

$$\alpha = \frac{1}{4}\frac{m_p / m_e}{a_0 / r_{pc}} = \frac{1}{4}\frac{r_{pc} m_p}{a_0 m_e} = \frac{1}{4}\frac{\mu}{\beta} = 0.007\ 297\ 352\ 569\ \underline{2858}\ \ (\text{rsu}\ 1.95 \text{ x } 10^{-12} \text{ vs } (1)) \tag{18}$$

where the underlined digits don't match those given in (1) - the computer 16 decimal digit precision limit didn't allow to reach an exact match -; note that (1) is full digit matched with the mean of (17) and (18). ρ was introduced as the proton charge to electron radius ratio with numeric value given by

$$\rho \equiv \frac{r_e}{r_{pc}} = 3.349\ \ 763\ \ 357\ \ 27689 \tag{19}$$

while β represents the Bohr H radius to the proton radius ratio with numeric value of

$$\beta \equiv \frac{a_0}{r_{pc}} = 62\ \ 904.754\ \ \ 018\ \ \ 5422 \tag{20}$$

(17) and (18) provide two new expressions to calculate μ and r_{pc} given by

$$\frac{\alpha\mu}{\rho} = \alpha \frac{m_p}{m_e}\frac{r_{pc}}{r_e} = 4 \quad (\text{rsu}\ \ 2.63\ \ \text{x}\ \ 10^{-12}) \tag{21}$$

and

$$\frac{\mu}{\alpha\beta} = \frac{1}{\alpha}\frac{m_p}{m_e}\frac{r_{pc}}{a_0} = 4 \quad (\text{rsu}\ \ 2.63\ \ \text{x}\ \ 10^{-12}) \tag{22}$$

And yet one more, their product after rearranging gives

$$\mu \frac{1}{\sqrt{\beta\rho}} = \frac{m_p}{m_e}\sqrt{\frac{r_{pc}^2}{a_0 r_e}} = 4 \quad (\text{rsu}\ \ 2.72\ \ \text{x}\ \ 10^{-12}) \tag{23}$$

But that's not all, dividing (21) by (22) we arrive to

$$\alpha^2 = \frac{r_e}{a_0} \tag{24}$$

which is equation (5) back to the stage! As it should be expected.

The numeric value for μ using (21), (22) and (23) are all the same and are given by

$$\mu = \frac{4}{\alpha}\frac{r_e}{r_p} = 4\alpha\frac{a_0}{r_p} = \frac{4}{r_p}\sqrt{a_0 r_e} = 1\ 836.152\ 673\ 43\underline{357} \quad (\text{rsu } 1.95 \text{ x } 10^{-12} \text{ wrt to } (2)) \tag{25}$$

where the underlined digits don't match the value given in (2). In the other side, the numeric value for r_{pc} using any one of (21), (22) and (23) is

$$r_{pc} = \frac{4}{\alpha\mu} r_e = \frac{4\alpha}{\mu} a_0 = \frac{4}{\mu}\sqrt{a_0 r_e} = 8.412\ 356\ 413\ 4\underline{445} \text{ x } 10^{-16} \text{ m } (\text{rsu } 1.95 \text{ x } 10^{-12} \text{ wrt to } (16)) \tag{26}$$

Equating (17) and (18), the following tidy expression for the five H atom parameters involved above is obtained

$$16\frac{\beta\rho}{\mu^2} = \left(\frac{4m_e}{r_{pc} m_p}\right)^2 r_e a_0 = 1 \quad (\text{rsu } 3.90 \text{ x } 10^{-12}) \tag{27}$$

and this provides

$$a_0 = \left(\frac{r_{pc}\mu}{4}\right)^2 \frac{1}{r_e} = 5.291\ 772\ 109\ 0\underline{0935} \text{ x } 10^{-11} \text{ m } (\text{rsu } 2.5 \text{ x } 10^{-12} \text{ wrt} (7)) \text{ and} \tag{28}$$

$$a_0 = \frac{r_{pc}\mu}{4\alpha} = 5.291\ 772\ 109\ 0\underline{0935} \text{ x } 10^{-11} \text{ m } (\text{rsu } 1.95 \text{ x } 10^{-12} \text{ wrt} (7)). \tag{29}$$

An equation, worth to compare with in regards to its definition and rsu level, is given by the a_0 equation provided in [1] and has the following numeric value

$$a_0 = \frac{\hbar}{\alpha m_e c} = 5.291\ 772\ 109\ 0\underline{4493} \text{ x } 10^{-11} \text{ m } (\text{rsu } 2.82 \text{ x } 10^{-12} \text{ wrt } (7)), \tag{30}$$

where (1) was used. Then, the rsu in (28) and (29) with the proton charge radius involved compare fairly well with the one in (30) whose definition involves two exact constants.

Expressions for the Wirkungsquantum of elementary particles.

Solving (30) for $a_0 m_e$ to substitute it in (18) and using (1) gives

$$\frac{2}{\pi}\alpha = \frac{r_{pc} m_p v_1}{h} = \frac{h_{p,1}}{h} \tag{31}$$

where $h_{p,1}$ is the proton quantum action or the magnitude of the proton quantum angular momentum for v_1 speed.

On the other side, (17) provides

$$\frac{\alpha}{4} = \frac{r_e m_e v_1}{r_{pc} m_p v_1} = \frac{h_{e,1}}{h_{p,1}} \tag{32}$$

where $h_{e,1}$ is the electron quantum action for v_1 speed. From (31) and (32), it follows that

$$h_{p,1} = \frac{2\alpha}{\pi} h \ \ \mathrm{J\,s} \quad ; \quad h_{p,1} = \frac{4}{\alpha} h_{e,1} \ \ \mathrm{J\,s} \quad \text{and} \quad h_{e,1} = \frac{\alpha^2}{2\pi} h \ \ \mathrm{J\,s} \tag{33}$$

Using (1), these equations are transformed into

$$h_{p,0} = \frac{2}{\pi} h \ \ \mathrm{J\,s} \quad ; \quad h_{p,1} = 4 h_{e,0} \ \ \mathrm{J\,s} \quad \text{and} \quad h_{e,0} = \frac{\alpha}{2\pi} h \ \ \mathrm{J\,s} \tag{34}$$

where $h_{e,0}$ and $h_{p,0}$ are the magnitudes of the electron and the proton quantum angular momenta at limit speed c, respectively. Additionally, the above expressions deliver

$$\alpha^3 = \frac{h_{e,1} h_{p,1}}{4\hbar^2} \quad ; \quad \alpha^2 = \frac{h_{e,0} h_{p,1}}{4\hbar^2} = \frac{h_{e,1} h_{p,0}}{4\hbar^2} \quad \text{and} \quad \alpha = \frac{h_{e,0} h_{p,0}}{4\hbar^2} \tag{35}$$

For the sake of completeness, equation (1) provides the above missing quantum action ratio which reads

$$\alpha \equiv \left\{ \frac{r_e m_e}{r_e m_e} \right\} \frac{v_1}{c} = \frac{h_{e,1}}{h_{e,0}} \tag{36}$$

Other calculation examples

An α-μ-r_{pc} expression involving also the Planck constant can be derived from the Rydberg constant times hc formula in NIST site [1] RyConst in eV which, after solving for h - in Js -, gives

$$h = \frac{\lambda_\infty}{2} \alpha^2 m_e c = 6.626\ \ 070\ \ 149\,98711\ \ \mathrm{x}\,10^{-34}\ \ \mathrm{J\,s}\ \ (\text{NOT exact!}) \tag{31}$$

where λ_∞, the Rydberg wavelength unit, is given as

$$\lambda_\infty = \frac{1}{R_\infty} = 91.\ 126\ \ 705\ \ 057\ \ 9021\ \ \mathrm{x}\,10^{-9}\ \ \mathrm{m/cy}\ \ (\mathrm{rsu}\ 1.9\,\mathrm{x}\,10^{-12}) \tag{32}$$

where R_∞ is the Rydberg constant R∞ NIST .

Using (17) to substitute one alpha factor in (31), this h-α-μ-r_{pc} 13-digit match equation is obtained

$$h = \frac{2r_e m_e}{\mu}\frac{\lambda_\infty}{r_{pc}}\alpha c = 6.626\ 070\ 15\ 00000\ \underline{4}\ \text{x}\ 10^{-34}\ \text{J s}\ (\text{rsu } 5.16\ \text{x}\ 10^{-15}) \tag{33}$$

This expression further confirms that the proposed proton charge radius is a good choice for performing low uncertainty parameter calculations.

Finally, the following all H radii expression can be derived using the above equations

$$\frac{1}{\mu\alpha^2} = \frac{1}{16\pi}\frac{1}{r_e / r_{pc}}\frac{1}{a_0 / \lambda_\infty} = \frac{1}{16\pi}\frac{1}{\rho}\frac{1}{\eta} = 10.2272895476\ 624 \approx \frac{32}{\pi} \tag{34}$$

where the following definition was introduced

$$\eta \equiv \frac{a_0}{\lambda_\infty} \tag{35}$$

An excellent review on regards to the development history of this $\mu\alpha^2$ relation is given by Helgen Kragh [12].

Conclusions

Using a straightforward derivation of two fine-structure constant physical definitions, analytic expressions for the proton charge radius and the proton to electron mass ratio were obtained.

In order to reduce the relative standard uncertainty level in α calculations using the obtained proton charge radius expressions, its currently accepted numeric value was modified by a small factor extracted along the used derivation procedure of the new definitions for α.

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Data availability

The data that support the findings of this study are available at the National Institute of Standards and Technology (NIST) https://physics.nist.gov/cuu/Constants/index.html .

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