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Uniform convergence analysis of finite difference approximations for general singular perturbed problem on adaptive grids

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Abstract

In this paper we consider a more general singular perturbation problem, that is, $-\varepsilon u''(x)-a(x)u'(x)+b(x)u(x)=f(x) \quad (0<\varepsilon\ll 1)$ on an adaptive grid. The mesh is constructed adaptively by equidistributing a monitor function based on the arc-length of the approximated solutions. Our analysis provide insight into the convergence behaviour on such mesh, and the posterior error estimates of piecewise linear interpolation about the approximate solution is investigated and an ε-uniform error estimate for the first-order upwind discretization of

general singular perturbed problem is derived at last. We extend the relevant results of the document to a more general case.

1. Introduction

This paper considers the numerical approximation of the general singular perturbed two point boundary value problem:

$$\left.\begin{aligned} &Tu(x) := -\varepsilon u''(x) - a(x)u'(x) + b(x)u(x) = f(x), x \in (0,1), \\ &u(0) = u_0,\ u(1) = u_1, \end{aligned}\right\} \qquad (1.1)$$

where $0<\varepsilon \ll 1$ is a small positive parameter and $u_0, u_1 \in \mathbb{R}$ are given constants. It is also assumed that $a(x), b(x), f(x) \in C^1[0,1]$, and there exist constants $\alpha, \bar{\alpha}$ and $\beta, \bar{\beta}$ for $\forall x \in [0,1]$ such that

$$0 < \alpha \le a(x) \le \bar{\alpha}, |a'(x)| \le \bar{\alpha} \qquad (1.2)$$

$$0 < \beta \le b(x) \le \bar{\beta}, |b'(x)| \le \bar{\beta}. \qquad (1.3)$$

It is well known that the problem (1.1) has a ε-related boundary layers [1]. For general coefficient functions *a*(*x*), *b*(*x*), *f*(*x*), and especially for nonlinear two-point BVPs, the width of the boundary layer is unknown. It is difficult to approximate efficiently by the most numerical methods especially using uniform mesh. The more widely applicable and better approach is to use an adaptive moving grid which based on equidistribution. A commonly used technique for determining the grid points is to require that they equidistribute a positive function of the numerical solution over the domain. For singular perturbation problems, an obvious choice of adaptively criterion is the solution gradient. There have been some results for adaptive mesh approach to the solution of the singularly perturbed problem (1.1) with $b(x) = 0$, $f(x) = 0$ see, e.g., [2-10]. More research on adaptive mesh methods can be referred to [11-17]. In our paper [17], we have weakened the mesh restriction and simplify the proofs of the stability and uniqueness by using the maximum principle. In this paper the posterior error estimates of piecewise linear interpolation about the approximate solution is derived, and it is different from the paper [17] which using piecewise quadratic interpolation of the approximate solution. This is an improvement for us. Finally we obtain an ε-uniform error estimate for the first-order upwind discretization of the general singular perturbed boundary value problem on an adaptive mesh. The research in this paper perfects the theoretical system for solving generalized singular perturbation problems, as well as extends and develops the work of [17].

Below we briefly introduce our numerical scheme that will be investigated in our work. Let

$$\Omega_N = \{x_i \mid 0 = x_0 < x_1 < x_2 < \ldots < x_N = 1\}$$

be an arbitrary nonuniform mesh on [0,1]. On Ω_N we discretize (1.1) as follows:

$$\left.\begin{array}{l} T^N u_i^N := -\varepsilon D D^- u_i^N - a_i D^+ u_i^N + b_i u_i^N = f_i, for 1 \le i \le N-1 \\ u_0^N = u_0, u_N^N = u_1 . \end{array}\right\} \quad (1.4)$$

where the operators used are given by

$$D^- w_i = \frac{w_i - w_{i-1}}{h_i},\ D^+ w_i = \frac{w_{i+1} - w_i}{h_{i+1}},\ D w_i = \frac{w_{i+1} - w_i}{\hbar_i}$$

$$h(i) = x_i - x_{i-1},\ \ \hbar(i) = \frac{h_i + h_{i+1}}{2}$$

A monitor function $M(x)$ is an arbitrary nonnegative function defined on Ω_N . A mesh x_i is said to equidistribute $M(x)$ if

$$\int_{x_{i-1}}^{x_i} M(x)dx = \frac{1}{N}\int_0^1 M(x)dx, 1 \le i \le N-1, \quad (1.5)$$

If N is given, the efficiency of an equidistributing mesh depends on the choice of $M(\cdot)$. The monitor function $M(x)$ in this paper is

$$M(x) = \sqrt{1 + \left[(u^N(x))'\right]^2}, \quad (1.6)$$

where $u^N(x)$ is piecewise linear interpolant through the knots (x_i, u_i^N) , and note that

$$(u^N(x))' = D^- u_i^N, x \in I_i = (x_{i-1}, x_i), 1 \le i \le N-1$$

Then, the equation (1.5) can be transformed into the discretization form

$$h_i M_i = \frac{1}{N}\sum_{j=1}^{N}(h_j M_j),\ \ 1 \le i \le N-1, \quad (1.7)$$

where

$M_i := \sqrt{1+[D^{-}u_i^N]^2}$.

The equation (1.7) also can be written as the following equidistribution mesh equations,

$$\left.\begin{aligned}&(x_{i+1}-x_i)^2+(u_{i+1}^N-u_i^N)^2=(x_i-x_{i-1})^2+(u_i^N-u_{i-1}^N)^2, 1\le i\le N-1,\\&u_0^N=u_0, u_N^N=u_1.\end{aligned}\right\} \tag{1.8}$$

With the definitions of equidistribution mesh and monitor function, the adaptive equidistribution algorithm used to approximate the solution of the problem (1.4) (see [17]).

2. Several useful conclusions about the solution of discrete problem

In our paper [17], we use the adaptive mesh equidistribution algorithm to construct a mapping and prove the existence, stability and uniqueness for the solution of the equidistribution problem by the fixed-point theorem and the maximum principle. We will also give the convergence result for this adaptive iteration algorithm about piecewise linear interpolant of the solution of (1.4).

In this section we will give the existence theorem, stability-uniqueness theorem and a useful corollary of the solution for the discrete problem, which has been proved in [17].

Theorem 2.1 (Existence theorem**).** For $0<\varepsilon\le 1$ and N>0, the equidistribution problem (1.4) and (1.8) has a solution; i.e., there exists a mesh that equidistributes the monitor function along the piecewise linear interpolant to the solution of (1.4).

Theorem 2.2 (Stability-uniqueness theorem**).** Let T^N be as in (1.4), for the $\{u_i^N\}$ defined on an arbitrary mesh $\{x^i\}$, we have

$$\max_{1\le i\le N-1}\left|u_i^N\right| \le \max\left\{\left|u_0^N\right|, \left|u_N^N\right|\right\} + C\max_{1\le i\le N-1}\left|T^N u_i^N\right|. \tag{2.1}$$

The constant C depends on the coefficients of T only.

Corollary 2.3. Let u(x) be the solution of (1.1), u_i^N be the solution of (1.4) on an arbitrary nonuniform mesh and $u^N(x)$ be its piecewise linear interpolation. Then we have

$$\left\|u^N(x)-u(x)\right\|_\infty \le C_1\left\|Tu^N(x)-Tu(x)\right\|_* . \tag{2.2}$$

where $\|v\|_\infty = ess\sup_{x\in[0,1]}|v(x)|$, $\|v\|_* = \min_{V:V'=v}\|V(x)\|_\infty = \min_{C\in R}\left\|\int_x^1 v(s)ds + C\right\|_\infty$.

3. Posterior error estimation and discretization accuracy

3.1. First-order a posterior error estimation

We now derive a posterior error estimation for solution of (1.4) and its piecewise linear interpolant on an arbitrary nonuniform mesh.

Theorem 3.1. Let u(x) be the solution of (1.1), u_i^N be the solution of (1.4) on an arbitrary nonuniform mesh and $u^N(x)$ be its piecewise linear interpolant. Then we have

$$\|u^N(x) - u(x)\|_\infty \le C \max_{1\le i\le N} \sqrt{(u_i^N - u_{i-1}^N)^2 + h_i^2}, \qquad (3.1)$$

where the constant C depends on $a(x), b(x), f(x)$ and Lipschitz constants L.

Proof. Let

$$\begin{aligned} &Bv(x) := \varepsilon v(x)' + a(x)v(x) + \int_x^1 a'(s)v(s)ds + \int_x^1 b(s)v(s)ds, \\ &F(x) := \int_x^1 f(s)ds, \\ &B^N v_i^N := \varepsilon D^- v_i^N + a_i v_i + \sum_{k=i}^{N-1} h_{k+1} D^+ a_k v_k + \sum_{k=i}^{N-1} h_{k+1} b_k v_k, \\ &F_i^N := \sum_{k=i}^{N-1} h_{k+1} f_k. \end{aligned} \qquad (3.2)$$

Note that

$$Tv(x) = -(Bv(x))', f(x) = -F'(x), \qquad (3.3)$$

Then

$$(Bu(x) - F(x))' = 0, x \in (0,1). \qquad (3.4)$$

So we have

$$B^N u_i^N - F_i^N = -m, 1 \le i \le N-1, \qquad (3.5)$$

where m is a constant. It follows from (3.3) that

$$Tu^N(x) - Tu(x) = Tu^N(x) - f(x) = -(Bu^N(x) - F(x) + c)' = (-Bu^N(x) + F(x) - c)'. \quad (3.6)$$

Using (3.5) and (3.6) gives

$$\left\|Tu^N(x) - Tu(x)\right\|_* = \min_{c\in R} \left\|Bu^N(x) - F(x) + c\right\|_\infty \le \left\|Bu^N(x) - F(x) + m\right\|_\infty. \quad (3.7)$$

Combining (3.5) and (3.7) we derive

$$Bu^N(x) - F(x) + m = Bu^N(x) - B^N u_i^N + F_i^N - F(x).$$

Since $(u^N(x))' = D^- u_i^N$ for $x \in I_i, 1 \le i \le N,$ then we conclude that

$$\begin{aligned} Bu^N(x) - B^N u_i^N &= \varepsilon(u^N(x))' + a(x)u^N(x) + \int_x^1 a'(s)u^N(s)ds + \int_x^1 b(s)u^N(s)ds \\ &\quad - \varepsilon D^- u_i^N - a_i u_i^N - \sum_{k=i}^{N-1} h_{k+1} D^+ a_k u_k^N - \sum_{k=i}^{N-1} h_{k+1} b_k u_k^N \\ &= \int_{x_i}^1 a'(s)u^N(s)ds - \int_x^{x_i} a(s)(u^N(s))'ds + \int_x^1 b(s)u^N(s)ds \\ &\quad - \sum_{k=i}^{N-1} h_{k+1} D^+ a_k u_k^N - \sum_{k=i}^{N-1} h_{k+1} b_k u_k^N. \end{aligned} \quad (3.8)$$

It's easy to derive

$$\begin{aligned} &\left|\int_{x_k}^{x_{k+1}} a'(s)u^N(s)ds - h_{k+1} D^+ a_k u_k^N\right| \\ &= \left|h_{k+1} a'(\xi)u^N(\xi) - h_{k+1} a'(\xi)u_k^N + h_{k+1} a'(\xi)u_k^N - h_{k+1} D^+ a_k u_k^N\right| \\ &\le \left|h_{k+1} a'(\xi)u^N(\xi) - h_{k+1} a'(\xi)u_k^N\right| + \left|h_{k+1} a'(\xi)u_k^N - h_{k+1} a'(\eta)u_k^N\right| \\ &\le \|a'\|_\infty h_{k+1} \left|u^N(\xi) - u_k^N\right| + h_{k+1} \left|u_k^N\right| \left|a'(\xi) - a'(\eta)\right| \\ &\le C\|a'\|_\infty h_{k+1} \left|u_{k+1}^N - u_k^N\right| + CLh_{k+1}^2, \end{aligned} \quad (3.9)$$

where $\xi, \eta \in (x_k, x_{k+1})$. It is similar to obtain the bound

$$\left|\int_{x_k}^{x_{k+1}} b(s)u^N(s)ds - h_{k+1} b_k u_k^N\right| \le C\|b\|_\infty h_{k+1} \left|u_{k+1}^N - u_k^N\right| + Ch_{k+1}^2, \quad (3.10)$$

Then we have

$$\begin{aligned} \left|\int_{x_i}^1 a'(s)u^N(s)ds - \sum_{k=i}^{N-1} h_{k+1} D^+ a_k u_k^N\right| &\le \sum_{k=i}^{N-1} \left|\int_{x_k}^{x_{k+1}} a'(s)u^N(s)ds - h_{k+1} D^+ a_k u_k^N\right| \\ &\le C\|a'\|_\infty \max_{0\le i\le N-1} \left|u_{i+1}^N - u_i^N\right| + CL \max_{0\le i\le N-1} h_{i+1}, \end{aligned} \quad (3.11)$$

$$\left|\int_{x_i}^{1} b(s)u^N(s)ds - \sum_{k=i}^{N-1} h_{k=1} b_k u_k^N\right| \le \sum_{k=i}^{N-1}\left|\int_{x_k}^{x_{k+1}} b(s)u^N(s)ds - h_{k+1}b_k u_k^N\right| \tag{3.12}$$
$$\le C\|b\|_\infty \max_{0\le i\le N-1}\left|u_{i+1}^N - u_i^N\right| + C\max_{0\le i\le N-1} h_{i+1.}$$

Moreover, we can obtain

$$\left|\int_{x}^{x_i} a(s)(u^N(s))'ds\right| \le \int_{x_{i-1}}^{x_i}\left|a(s)(u^N(s))'\right|ds \le \|a\|_\infty \left|u_i^N - u_{i-1}^N\right|, \tag{3.13}$$

$$\left|\int_{x}^{x_i} b(s)u^N(s)ds\right| = \left|\int_{x}^{x_i} b(s)h_i^{-1}\left[u_i^N(s-x_{i-1}) + u_{i-1}^N(x_i - s)\right]ds\right|$$
$$\le \int_{x}^{x_i}\left|b(s)h_i^{-1}\right|\left[\left|u_i^N(s-x_{i-1})\right| + \left|u_{i-1}^N(x_i-s)\right|\right]ds \tag{3.14}$$
$$\le C\|b\|_\infty h_{i.}$$

Hence, using (3.8)-(3.14) we have

$$\left|Bu^N(x) - B^N u_i^N\right| = \left|\int_{x_i}^{1} a'(s)u^N(s)ds - \sum_{k=i}^{N-1} h_{k+1} D^+ a_k u_k^N - \int_{x}^{x_i} a(s)(u^N(s))'ds\right.$$
$$\left. + \int_{x}^{x_i} b(s)u^N(s)ds + \int_{x_i}^{1} b(s)u^N(s)ds - \sum_{k=i}^{N-1} h_{k+1} b_k u_k^N\right| \tag{3.15}$$
$$\le C(\|a'\|_\infty + \|a\|_\infty + \|b\|_\infty)\max_{0\le i\le N-1}\left|u_{i+1}^N - u_i^N\right| + (CL + C + C\|b\|_\infty)\max_{0\le i\le N-1} h_{i+1}$$
$$\le C\max_{0\le i\le N-1}\left|u_{i+1}^N - u_i^N\right| + C\max_{0\le i\le N-1} h_{i+1.}$$

Furthermore, we have the bound for the $F_i^N - F(x)$

$$\left|\int_{x}^{x_i} f(s)ds\right| \le \|f\|_\infty h_i, \left|h_{k+1}f_{k+1} - \int_{x_k}^{x_{k+1}} f(s)ds\right| \le h_{k+1}\left|f_{k+1} - f(\xi)\right| \le \|f'\|_\infty h_{k+1}{}^2,$$
$$\Rightarrow \left|F_i^N - F(x)\right| = \left|\sum_{k=i}^{N-1} h_{k+1} f_x - \int_{x_i}^{1} f(s)ds - \int_{x}^{x_i} f(s)ds\right| \tag{3.16}$$
$$\le (\|f\|_\infty + \|f'\|_\infty)\max_{0\le i\le N-1} h_{i+1} \le C\max_{0\le i\le N+1} h_{i+1.}$$

Finally from corollary 2.3, (3.7), (3.15) and (3.16) we obtain

$$
\begin{aligned}
\left\|u^N(x)-u(x)\right\|_\infty &\le C_1\left\|Tu^N(x)-Tu(x)\right\|_* \\
&\le C_1\left\|Bu^N(x)-F(x)+m\right\|_\infty \\
&= C_1\left\|Bu^N(x)-B^N u_i^N+F_i^N-F(x)\right\|_\infty \\
&\le C\max_{0\le i\le N-1}\left|u_{i+1}^N-u_i^N\right|+C\max_{0\le i\le n-1}h_{i+1} \\
&\le \sqrt{2}C\max_{1\le i\le N}\sqrt{(u_i^N-u_{i-1}^N)^2+h_i^2} \\
&\le C\max_{1\le i\le N}\sqrt{(u_i^N-u_{i-1}^N)^2+h_i^2}.
\end{aligned}
\tag{3.17}
$$

which completes the proof.

3.2. Accuracy of the fully discretized solution

Next we analyse the accuracy of the fully discretized solution, that establish the first-order ε-independent convergence rate for the difference scheme (1.4) and equidistribution mesh equation (1.8).

Theorem 3.2. Let u(x) be the solution of (1.1), u_i^N be the solution of (1.4) computed on the equidistributing mesh $\{x_i\}$ satisfying (1.8), and $u^N(x)$ be the piecewise line interpolation function. Then we have

$$\max_{0\le x\le 1}\left|u(x)-u^N(x)\right|\le CN^{-1}, \tag{3.18}$$

where the C is independent of ε and N.

Proof. Let

$$L^N=\sum_{i=1}^N l_i^N=\sum_{i=1}^N h_i\sqrt{1+(D^- u_i^N)^2}\ge\sum_{i=1}^N h_i=1. \tag{3.19}$$

It follows from (1.7) that

$$l_i^N := h_i\sqrt{1+(D^- u_i^N)^2}=\frac{L^N}{N}, 1\le i\le N. \tag{3.20}$$

For (3.20), we can prove that (see [17])

$$l_i^N\le\frac{C_2}{N}, \tag{3.21}$$

Using theorem 3.1, (3.20) and (3.21), we have

$$
\begin{aligned}
\left|u(x)-u^N(x)\right| &\le \left\|u^N(x)-u^N(x)\right\|_\infty \le C\max_{0\le x\le 1}\sqrt{(u_i^N-u_{i-1}^N)^2+h_i^2} \\
&= C\max_{1\le i\le N} h_i\sqrt{1+(D^-u_i^N)^2} = C\max_{1\le i\le N} l_i^N \qquad (3.22)\\
&\le C_2CN^{-1} \le CN^{-1}.
\end{aligned}
$$

This complete the proof of this theorem.

4. Numerical experiments

We consider the following general inhomogeneous singular perturbation test problem:

$$
\begin{cases}
-\varepsilon u''(x) - xu'(x) + 2u(x) = f(x), \quad x\in(0,1), \\
u(0)=0, \quad u(1)=1.
\end{cases} \qquad (4.1)
$$

where $f(x)=2\text{-}2\varepsilon e^{-1/\sqrt{\varepsilon}}-\left(1+x/\sqrt{\varepsilon}\right)e^{-x/\sqrt{\varepsilon}}$. The exact solution of (4.1) is $u(x)=x^2e^{-1/\sqrt{\varepsilon}}-e^{-x/\sqrt{\varepsilon}}+1$.

The numerical experiments in Figure 1 show that the adaptive grid method is much effective in solving the general singular perturbation problem.

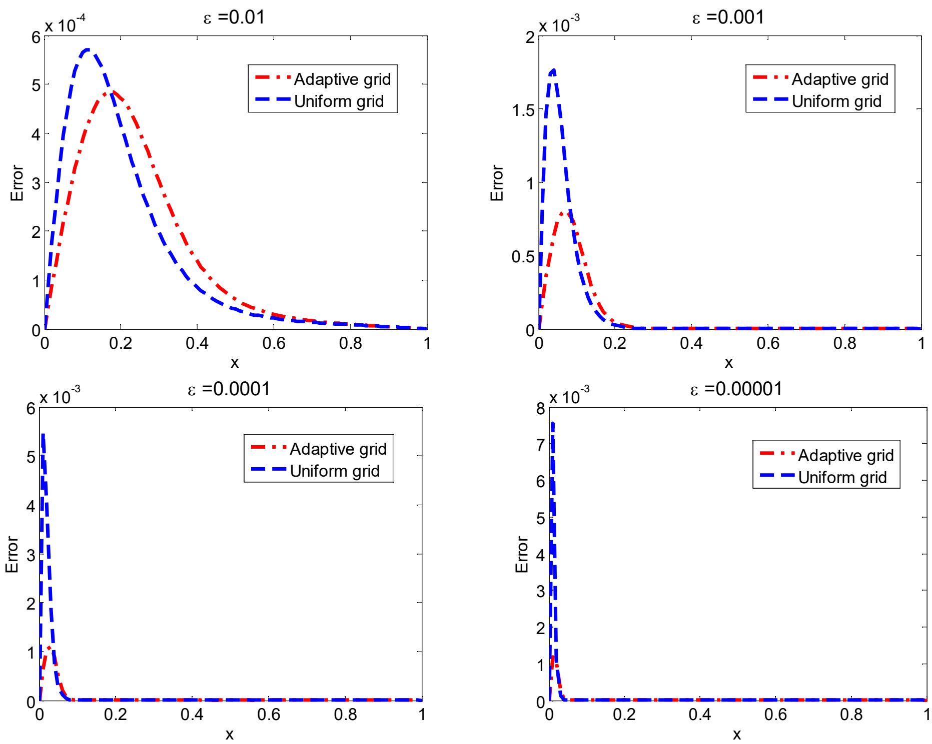

Figure 1. The errors of the algorithms on uniform and adaptive grids with N=1000.

5. Conclusions

There is an improvement in this paper that the posterior error estimates of piecewise linear interpolation of the approximate solution for the general singularly perturbed problem is obtained. At last we obtain a first-order ε-independent convergence rate for the numerical algorithm defined on the adaptive equidistribution mesh and an estimation for the accuracy of fully discretized solution. The research in this paper extends and develops the work of [17]. Numerical results confirmed the theoretical error estimation and the efficiency of the adaptive mesh.

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