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Structural Presentation of Some Concepts in Metric Spaces and Topological Spaces

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Abstract

The concept of metric space and topological space grew out of the study of real line and Euclidean space and study of continuous function on these spaces. In this Paper, we will try to discuss some concepts in metric spaces and topological spaces whose meanings are same but presented by the different ways. We will discuss open sets, closed sets, continuity and connectedness in the reference to metric space and topological space.

Keywords: Metric Space, Topological Space and Compactness

1. Introduction

Topology is qualitative mathematics, that is, it is mathematics without numbers. Topology is concerned with those intrinsic qualitative properties of spatial configuration that are independent of size, location and shape. In this paper, we define metric space and topological space and study a number of ways of constructing a metric and topology on a set so as to make it into metric space and topological space. We also consider some of elementary concepts associated with metric spaces and topological spaces. Specially, open sets, closed sets continuity and compactness.

2. Definition and Examples of Metric Spaces

In the theory of functions of real variable the notion of distance plays a vital role in formulating the definition of convergence, continuity and differentiability. Metric spaces are sets in which there is defined a notion of distance between pair of points and they provide the general setting in which we study convergence and continuity. The concept of metric spaces was formulated in 1906 by M. Frechet.

Definition 1: Let *X* be a nonempty set. A metric on *X* is a real valued function $d: X \times X \to R$ which satisfies the following properties:

1. $d(x, y) \geq 0, \forall x, y \in X$;
2. $d(x, y) = 0 \Leftrightarrow x = y, \forall x, y \in X$;
3. $d(x, y) = d(y, x), \forall x, y \in X$ (Symmetry)
4. $d(x, y) \leq d(x, z) + d(z, y) \forall x, y \in X$ (Triangle Inequality).

A metric *d* is also called a distance function, and the non-negative real number $d(x, y)$ is to be thought of as the distance between *x* and *y*.

A metric space is a non-empty set *X* equipped with a metric *d* on *X* and is denoted by the pair (*X, d*), or simply *X*.

Different metrics can be defined on a single non-empty set and this gives rise to distinct metric spaces.

Example 1. The function $d: R \times R \to R$ defined by $d(x, y) = |x - y|$, $\forall x, y \in R$ is a metric on the set of all real numbers *R*.

Since for $x, y, z \in R$, we have

$$d(x, y) = |x - y| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$$

The number $d(x, y)$ is of course, the usual distance between the points x, *y* on the real line. Therefore *d* is referred to as the usual metric on *R* and (*R, d*) is called the **usual metric space**.

Example 2. Let *X* be an arbitrary non-empty set. The function *d* defined by

$$d(x,y) = \begin{cases} 1, if\ x \neq y \\ 0, if\ x = y, \end{cases} \forall x, y\ \in X$$

is a metric on *X* and is called the discrete or trivial metric on *X*, and (*X, d*) is called the discrete metric space or trivial metric space.

Example 3. The function *d* defined by

$$d(x,y) = (\sum_{i=1}^{n} (x_i - y_i)^2)^{\frac{1}{2}}, \forall x = (x_1, x_2, \dots, x_n), y\ = (y_1, y_2, \dots, y_n) \in R^n$$

is a metric on R^n, where R^n is the set of all ordered n-tuples and *d* is called the Euclidean metric, also the metric space (R^n, d) is called Euclidean metric space.

Example 4. Let (*X, d*) be any metric space. Show that d_1 defined by

$$d_1(x,y) = \frac{d(x,y)}{1 + d(x,y)}, \forall x, y\ \in X$$

is a metric on *X*.

For the triangle inequality we proceed as follows:

Using the triangle inequality for the metric d, we have for all $x, y\ , z \in X$.

$$d(x,y) \leq d(x,z) + d(z,y)$$
$$1 + d(x,y) \leq 1 + d(x,z) + d(z,y)$$

$$1 - \frac{1}{1 + d(x,y)} \leq 1 - \frac{1}{1 + d(x,z) + d(z,y)}$$

$$\frac{d(x,y)}{1 + d(x,y)} \leq \frac{d(x,z) + d(z,y)}{1 + d(x,z) + d(z,y)}$$

$$\leq \frac{d(x,z)}{1 + d(x,z)} + \frac{d(z,y)}{1 + d(z,y)}$$

$$d_1(x,y) \leq d_1(x,z) + d_1(z,y)$$

This shows that d_1 is a metric on X.

3. Definition and Examples of Topological Spaces

The definition of a topological space is now standard was a long time in being formulated. Many mathematicians proposed different definitions. The definition finally settled on may seem a bit abstract, but as you work through the various ways of constructing topological spaces, you will get a better feeling for what the concept mean.

Definition 2 : A topology on a set *X* is a collection τ of subsets of *X* having the following Properties:

1. $\emptyset$ and X are in τ.
2. The union of the elements of any subcollection of τ is in τ.
3. The intersection of the elements of any finite subcollection of τ is in τ.

A set *X* for which a topology τ has been specified is called a topological space.

Properly speaking, a topological space is an ordered pair (*X*, τ) consisting of a set *X* and a topology τ on *X*, but we often omit specific mention of τ if no confusion will arise.

If *X* is a topological space with topology τ, we say that a subset *U* of *X* is an open set of *X* if *U* belongs to the collection τ. Using this terminology, one can say that a topological space is a set *X* with the collection of subsets of *X*, called open sets such that empty set and *X* are both open and such that arbitrary unions and finite intersections of open sets are open.

Example 5: Let $X = \{a, b, c\}$ be a set of three elements. There are many topologies on *X*. Some of which are defined as follows:

$$T_1 = \{\emptyset, X\}$$

$$T_2 = \{\emptyset, X, \{a\}, \{a, b\}\}$$

$$T_3 = \{\emptyset, X, \{a\}\}$$

$$T_4 = \{\emptyset, X, \{a, b\}\}$$

$$T_5 = \{\emptyset, X, \{a\}, \{c, b\}\}$$

$$T_6 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$$

$$T_7 = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$$

$$T_8 = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$$

Example 6. If *X* is any set, the collection of all subsets of *X* is a topology on *X* shown above by T_8 is called the discrete topology. The collection consisting of *X* and empty set only shown above by T_1 is also a topology on *X*, we shall call it the indiscrete topology or trivial topology.

4. Open and Closed Sets in Metric Spaces and Topological Spaces

Let (X, d) be any metric space, and $a \in X$. Then for any $r > 0$, the set

$$S_r(a) = \{x \in X : d(x, a) < r\}$$

is called an open sphere or open ball of radius *r* centered at *a*.

The set

$$S_r[a] = \{x \in X : d(x, a) \leq r\}$$

is called a closed sphere of radius *r* centered at *a*. It is clear that

$$S_r(a) \subset S_r[a]$$

for every $a \in X$ and for every $r > 0$.

In the metric space (R, d) of real numbers with usual metric *d*, the open sphere $S_r(a)$ is the open interval]*a-r, a+r*[, and the closed sphere $S_r[a]$ is the closed interval [*a-r, a+r*], where $a \in R$ and every $r > 0$.

Let (*X, d*) be a metric space and $a \in X$. A subset N_a of *X* is called a neighborhood of a point $a \in X$, if there exists an open sphere $S_r(a)$ centered at '*a*' and contained in N_a; that is $S_r(a) \subset N_a$, for some $r > 0$.

Definition 3: A subset *G* of a metric space (X, d) is said to be open in *X* with respect to the metric *d*, if *G* is a neighborhood of each of its points. In other word, if for each $a \in G$, there is a $r > 0$ such that $S_r(a) \subset G$.

Every open sphere is an open set.

In any metric space (X, d)

i. The union of an arbitrary family of open sets is open.
ii. The intersection of a finite number of open sets is open.

Definition 4: Let *A* be any subset of a metric space (X, d). A point '*a'* of *X* is called limit point of

A if every open sphere centered at '*a'* contains at least one member of *A* other than '*a'*; that is

$$S_r(a) \cap (A - \{a\}) \neq \emptyset, \forall\, r > 0$$

Definition 5: A subset *F* of a metric space (X, d) is said to be closed if *F* contains all its limit points.

In every metric space (X, d), a subset *F* of *X* is closed if and only if its complement in *X* is open and also every closed sphere is a closed set.

A subset *A* of a topological space *X* is said to be closed if the set (*X-A*), the complement of *A* in *X* is open.

Example 7: Show that every set in a discrete metric space (X, d) is open.

Let *G* be any non-empty subset of the discrete metric space (*X, d*) and *x* be any point of *G*. Then the open sphere $S_r(x)$ with $r \leq 1$ is the singleton set $\{x\}$ which is contained in *G*. that is each point of *G* is the centre of some open sphere contained in *G*. In particular each singleton set is open.

Example 8: Show that on the real line with usual metric the singleton set $\{x\}$ is not open.

For the metric space (R, d) each open sphere $S_r(x)$ is the bounded open interval $]a\text{-}r, a+r[$ and for no value of *r* this sphere is contained in $\{x\}$. Hence $\{x\}$ is not open in *R*.

Example 9: The subset [a, b] of the topological space *R* is closed because its complement

$$R - [a, b] = (-\infty, a) \cup (b, \infty)$$

Is open. Similarly, $[a, +\infty)$ is closed, because its complement $(-\infty, a)$ is open. The subset $[a, b)$ of *R* is neither open nor closed.

Example 10: In the discrete topology on *X*, every set is open; it follows that every set is closed as well.

5. Continuity in Metric Spaces and Topological Spaces

Definition 6: Let (X, d_1) and (Y, d_2) be any two metric spaces. A function $f: X \to Y$ is said to be continuous at a point *a* of *X*, if for given $\epsilon > 0$, there exists a $\delta > 0$, such that

$$d_2\big(f(x), f(a)\big) < \epsilon, \text{ whenever } d_1(x, a) < \delta.$$

Equivalently, for each open sphere $S_\epsilon(f(a))$ centered at $f(a)$ there is an open sphere $S_\delta(a)$ centered at *a* such that

$$f(S_\delta(a)) \subseteq S_\epsilon(f(a))$$

The function $f: X \to Y$ is said to be continuous, if it is continuous at each point of *X*.

Example 11: If (X, d) is a discrete metric space then every function $f: X \to Y$ is continuous on *X*.

For any $a \in X$, if we choose $\delta < 1$. Then $S_\delta(a) = \{a\}$ and so

$$f\big(S_\delta(a)\big) = f(a) \subseteq S_\epsilon(f(a))$$

holds for each $\epsilon > 0$.

Example 12: If (X, d_1) and (Y, d_2) be any two metric spaces; then the constant function $f: X \to Y$ is continuous on *X*.

Definition 7: Let *X* and *Y* be topological spaces. A function $f: X \to Y$ is said to be continuous if for each open subset *V* of *Y*, the set $f^{-1}(V)$ is an open subset of *X*.

Recall that $f^{-1}(V)$ is the set of all points *x* of *X* for which $f(x) \in V$; it is empty if *V* does not intersect the image set $f(X)$ of *f*.

Continuity of a function depends not only upon the function *f* itself, but also on the topologies specified for its domain and range. If we wish to emphasize this fact, we can say that f is continuous relative to specific topologies on *X* and *Y*.

Some equivalent theorems in metric space and topological space:

Theorem 1: (X, d_1) and (Y, d_2) be any two metric spaces, then function $f: X \to Y$ is continuous if and only if $f^{-1}(V)$ is open in *X*, whenever *V* is open in *Y*.

Proof: We first assume that *f* is continuous. If *V* is any open subset in *Y*, we shall show that $f^{-1}(V)$ is open in *X*. If $f^{-1}(V) = \emptyset$, it is open; so we assume that $f^{-1}(V) \neq \emptyset$. Let $x \in f^{-1}(V)$. Then $f(x) \in V$ and since *V* is open there exists an open sphere $S_\epsilon(f(x))$ such that $S_\epsilon(f(x)) \subseteq V$, for some $\epsilon > 0$.

Now by definition of continuity, there exists an open sphere $S_\delta(x)$ such that

$$f(S_\delta(x)) \subseteq S_\epsilon(f(x))$$

for $\delta > 0$. But

$S_\epsilon(f(x)) \subseteq V \Longrightarrow S_\delta(x) \subseteq f^{-1}(V)$.

Therefore $f^{-1}(V)$ is open.

Now we assume that $f^{-1}(V)$ is open in *X*, whenever *V* is open in *Y*, and show that *f* is continuous. Let *x* be an arbitrary point in *X*, and let $\epsilon > 0$ be given. Let $S_\epsilon(f(x))$ be an open sphere in *Y* centered at $f(x)$.

This open sphere is an open set, so its inverse is an open set which contains x.

i.e. $f^{-1}(S_\epsilon(f(x)))$ is open in X.

Since $x \in f^{-1}(S_\epsilon(f(x)))$, there exists a $\delta > 0$ such that

$$S_\delta(x) \subseteq f^{-1}(S_\epsilon(f(x))) \Rightarrow f(S_\delta(x)) \subseteq S_\epsilon(f(x))$$

Hence *f* is continuous at *x*. Since *x* was taken to be an arbitrary point of *X*. Hence *f* is continuous at every point of *X*.

Theorem 2 : Let *X* and *Y* be topological spaces. A function $f: X \to Y$ is continuous if and only if for each $x \in X$ and each neighbourhood *V* of $f(x)$, there is a neighborhood *U* of *x* such that $f(U) \subseteq V$.

Proof: Let $x \in X$ and let *V* be a neighborhood of $f(x)$. Then the set $U = f^{-1}(V)$ is a neighborhood of *x* such that $f(U) \subseteq V$.

Conversely, let *V* be an open set of *Y*; let *x* be a point of $f^{-1}(V)$. Then $f(x) \in V$, so that by hypothesis there is a neighourhood U_x of *x* such that $f(U_x) \subseteq V$. Then $U_x \subseteq f^{-1}(V)$. It follows that $f^{-1}(V)$ can be written as the union of the open sets U_x, so that it is open.

6. Compactness in Metric spaces and Topological spaces

The concept of compactness is an abstraction of an important property known as ' Heine Boral Property' possed of *R* which is closed and bounded. Heine Boral theorem states that if$I \subseteq R$ is a closed interval and family of open interval in *R* whose union contains *I* has a finite subfamily whose union contains *I*. Compactness is concerned with covering the sets by open sets. Before defining compactness we need the following definitions.

Definition 8: Let (X, d) be a metric space or a topological space (X, τ). A family of subsets $\{A_\alpha\}$ in *X* is

called a cover of any subset A of X if $A \subseteq \bigcup_{\alpha \in \Gamma} A_\alpha$, Γ is any non empty index set. If each A_α, $\alpha \in \Gamma$, is

open in *X*, then the cover $\{A_\alpha\}$ is called an open cover of A.

A subfamily $\{A_\alpha\}$ which itself is an open cover is called sub-cover of *A*. If the number of members in the subfamily is finite it is called a finite subcover of A.

Definition 9: A subset A of a metric space (X, d) or a topological space (X, τ) is said to be compact if every open cover of A admits of a finite subcover, i.e. for each family of open subsets $\{G_\alpha\}$ of X for which $\bigcup_{\alpha\in\Gamma} G_\alpha \supseteq A$, there exists a finite subfamily say $\{G_{\alpha_1}, G_{\alpha_2}, G_{\alpha_3}, ..., G_{\alpha_n}\}$ such that $A \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$.

A metric space (X, d) or a topological space (X, τ) is compact if X is compact, i.e., each family of open subsets $\{G_\alpha\}$ of X for which $\bigcup_{\alpha\in\Gamma} G_\alpha = X$, there exists a finite subfamily $\{G_{\alpha_1}, G_{\alpha_2}, G_{\alpha_3}, ..., G_{\alpha_n}\}$ such that $X = \bigcup_{i=1}^{n} G_{\alpha_i}$.

Illustrations:

i. Any closed interval with usual metric is compact.

ii. The discrete metric space (X, d) or a discrete topological space (X, τ), where X is a finite set, is compact.

iii. The space (R, d), where R is the set of reals and d is the usual metric is not compact, for the cover {]-n, n[: n is natural number} is such that $\bigcup_{i=1}^{n}]-i, i[= R$, but we do not have finite subcover.

Example 13. Prove that the open interval]0, 1[with usual metric is not compact. The family of open

intervals $\{]\frac{1}{n}, 1[: n = 2, 3, ...\}$ is such that $\bigcup_{n=2}^{\infty}]\frac{1}{n}, 1[=]0, 1[$. Therefore $\{]\frac{1}{n}, 1[: n = 2, 3, ...\}$

is an open cover of]0, 1[, which has no finite subcover.

Example 14. Let X be an infinite set with the discrete metric. Show that (X, d) is not compact. For each $x \in X$, $\{x\}$ is open in X. Also $\bigcup_{x\in X} \{x\} = X$.Therefore $\{\{x\}: x \in X\}$ is an open cover of X and since X is infinite; this open cover has no finite subcover.

Theorem 3: Every closed subset of a compact metric space is compact.

Proof: Let (X, d) be any compact metric space and F be any nonempty closed subset of X. We shall show that F is compact.

Let $\{G_\alpha, \alpha \in \Gamma\}$ be a family of open sets in X such that $\bigcup_{\alpha\in\Gamma} G_\alpha \supseteq F$. Then $(\bigcup_{\alpha\in\Gamma} G_\alpha) \cup (X - F)$ is an open cover of X and by compactness of X, it has a finite subcover, say $G_{\alpha_1}, G_{\alpha_2}, G_{\alpha_3}, ..., G_{\alpha_n}, X - F$. Therefore $(\bigcup_{i=1}^{n} G_{\alpha_i}) \cup (X - F) = X \Longrightarrow \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq F$. Hence F is compact.

Note: This theorem shows that each closed subset of a compact metric space is compact. On the real line the closed set N, set of natural numbers is not compact in R. Also the closed set F= [0, 1] is not compact in (X, d) where X=]0, 2] and d is the usual metric.

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