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Estimating the Overlapping Coefficient in the Case of Normal Distributions

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Abstract

Given that we have two independent random samples, each of which follows a normal distribution, the main objective of this paper is to estimate the overlapping Weitzman coefficient Δ. This coefficient is widely used and is defined as the area under two probability density functions. The proposed estimation technique is based on the rules of integral numerical approximation such as trapezoidal rules and Simpson's rules. Simulation results showed the effectiveness of the proposed technique over some of the methods found in the literature.

Keywords: Normal Distribution; Numerical Integration Methods; Maximum Likelihood Method; Relative Bias; Relative Mean Square Error.

1. Introduction

There are three overlapping (OVL) coefficients, namely Matusita (1955) coefficient (ρ), Morisita (1959) coefficient (λ) and Weitzman (1970) coefficient (Δ). These different coefficients represent the degree of similarity or closeness between two phenomena. Our main interest in this paper is to estimate the Weizmann coefficient Δ under the assumption that there are two normal distributions without using any constraints on the parameters of these distributions. Let $f_1(x)$ and $f_2(x)$ are two continuous probability density functions, the Weitzman OVL coefficient is defined as follows,

$$\Delta = \int min\{f_1(x), f_2(x)\}dx.$$

Noting that the values of Δ fall in the interval [0,1]. If its value is close to zero, it means that there is no common area between $f_1(x)$ and $f_2(x)$. On the other hand, if its value is close to 1, it indicates a perfect match for $f_1(x)$ and $f_2(x)$.

There are many applications of OVL coefficients, in particular the Weitzmann coefficient, which has been used in the fields of environment (Pianka, 1973), income (Gastwirth, 1975), and genetic (Federer et al., 1963).

In the literature, there are two methods to estimate OVL, the parametric method and the nonparametric method (see Eidous and Al-Talafheh, 2022 and Eidous and Ananbeh, 2024). Several authors considered the parametric method to estimate different OVL coefficients (see Eidous and Daradkeh, 2022 and Eidous and Al-Hayja'a, 2023a and the reference therein). Inman and Bradly (1989) derived the maximum likelihood (ML) estimator of Δ under the assumption that the two densities are normal with equal variances and different means. Under the normality assumption with equal variance, Reiser and Faraggi (1999) constructed a confidence interval for Δ. Eidous and Al Shourman (2022) used the numerical integral method to estimate Matusita overlapping measure in the case of two normal distributions. There are other studies in the literature to estimate the Weismann coefficient under the assumption of continuous distributions other than the normal distribution, such as Weibull, Gamma and Pareto distributions (See, Madhuri *et al*., 2001, Eidous and Al-Hayja'a 2023b and Wang and Tian, 2017).

It should be noted here that all the previously mentioned studies that used two-parameter distributions placing restrictions on the parameters of the distributions (i.e. the condition of equal location or equal scale or equal shape parameters). If there is some doubts about the validity of the model assumption or if the model assumption is difficult to be determine then the nonparametric method can be used instead of the parametric method. Some authors have studied the nonparametric method to make inferences about overlapping coefficients, (see for example, Clemons and Bradley, 2000 and Eidous and Al-Talafha, 2022).

The main contribution of this paper is to develop new estimators for Δ in the case of pair normal distributions without using any assumptions on the parameters of the pair distributions. This objective can be achieved by approximating the integral that appears in the formula of Δ and then estimating the obtained approximations using the ML method. Accordingly, this paper is organized as the following: Section (2) introduces the existing parametric estimators of Δ under pair normal distributions. In Section (3), some approximations of Δ based on numerical integration methods are given. The proposed estimation technique for Δ is given in Section (4) and three new estimators are derived. Section (5) gives a simulation study to investigate the properties of the proposed estimators and to compare them with the estimator of Δ that suggested by Eidous and Al-Daradkeh (2023).

2. Weitzman coefficient Δ for two normal distributions

Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$, where X_1 and X_2 are two independent random variables. Under the assumption that the two variances are equal (i.e. $\sigma_1^2 = \sigma_2^2 = \sigma^2$, say), Inman and Bradley (1989) derived the formula of Δ, which is given by

$$\Delta = 2\Phi\left(-\frac{|\mu_1 - \mu_2|}{2\sigma}\right),$$

where $\Phi(t)$ is the cumulative standard normal distribution at a point t. If $(X_{11,} X_{12}, \ldots . X_{1n_1})$ and $(X_{21,} X_{22}, \ldots . X_{2n_2})$ are two independent random samples taken from $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ respectively, then Inman and Bradley (1989) gave the maximum likelihood (ML) estimator of Δ, which is given by

$$\widehat{\Delta}_{IB} = 2\Phi\left(-\frac{|\overline{X}_1 - \overline{X}_2|}{2S}\right),$$

where $\overline{X}_i = \sum_{j=1}^{n_1} X_{1j}/n_i, (i = 1, 2)$ are the ML estimators of μ_1 and μ_2 respectively, and S is the square root of S^2, which is the maximum likelihood estimator of σ^2, given by

$$S^2 = \frac{\sum_{j=1}^{n_1} (X_{1j} - \overline{X}_1)^2 + \sum_{j=1}^{n_2} (X_{2j} - \overline{X}_2)^2}{n_1 + n_2}.$$

Now, under the assumption $\mu_1 = \mu_2 = (\mu$, say), Mulekar and Mishra (1994) derived the formula of Δ, which is given by

$$\Delta = \begin{cases} 1 - 2\Phi(b) + 2\Phi(\mathrm{C}b) & \text{if } 0 < C < 1 \\ 1 + 2\Phi(b) - 2\Phi(\mathrm{C}b) & \text{if } \quad \mathrm{C} \geq 1 \end{cases},$$

where $C = \sigma_1/\sigma_2$ and $b = \sqrt{-ln(C^2)/(1-C^2)}$. Their corresponding estimator of Δ is,

$$\widehat{\Delta}_{MM} = \begin{cases} 1 - 2\Phi(\hat{b}) + 2\Phi(\hat{C}\hat{b}) & \text{if } 0 < \hat{C} < 1 \\ 1 + 2\Phi(\hat{b}) - 2\Phi(\hat{C}\hat{b}) & \text{if } \quad \hat{C} \geq 1 \end{cases},$$

where $\hat{C} = \frac{n_2}{n_1} \frac{\sum_{i=1}^{n_1} (X_{1i}-\hat{\mu})^2}{\sum_{i=1}^{n_2} (X_{2i}-\hat{\mu})^2}$, $\hat{\mu} = \frac{n_1 \bar{X}_1 + n_2 \bar{X}_2}{n_1+n_2}$ and $\hat{b} = \sqrt{-ln(\hat{C}^2)/(1-\hat{C}^2)}$.

In order to be able to use any of the previous two estimators, the corresponding study to each of them requires some restrictions on the parameters of the two distributions. The first study assumes that the two scale parameters are equal, while the second study requires that the location parameters are equal. Without using these assumptions, neither the estimator $\widehat{\Delta}_{IB}$ nor $\widehat{\Delta}_{MM}$ can be used to estimate Δ**.** To overcome this problem and without using any assumptions about the distributions parameters, Eidous and Al-Daradkeh (2023) expressed the formula of Δ as follows,

$$\Delta = \frac{1}{2}\left[E\left(\frac{min\{f_1(X_1), f_2(X_1)\}}{f_1(X_1)}\right) + E\left(\frac{min\{f_1(X_2), f_2(X_2)\}}{f_2(X_2)}\right)\right].$$

and they gave the following estimator for Δ,

$$\hat{\Delta}_{ED} = \frac{1}{2}\left[\frac{1}{n_1}\sum_{j=1}^{n_1}\left(\frac{min\{f_1(X_{1j};\hat{\mu}_1,\hat{\sigma}_1^2),\ f_2(X_{1j};\hat{\mu}_2,\hat{\sigma}_2^2)\}}{f_1(X_{1j};\hat{\mu}_1,\hat{\sigma}_1^2)}\right)\right.$$
$$\left. + \frac{1}{n_2}\sum_{j=1}^{n_2}\left(\frac{min\{f_1(X_{2j};\hat{\mu}_1,\hat{\sigma}_1^2),\ f_2(X_{2j};\hat{\mu}_2,\hat{\sigma}_2^2)\}}{f_2(X_{2j};\hat{\mu}_2,\hat{\sigma}_2^2)}\right)\right]$$

where f_1 and f_2 are two normal distributions, such that,

$$f_i(X_{ij};\hat{\mu}_i,\hat{\sigma}_i^2) = \left(2\pi\hat{\sigma}_i^2\right)^{-1/2} e^{-\left(X_{ij}-\hat{\mu}_i\right)^2/2\hat{\sigma}_i^2},\ i = 1, 2\,.$$

Also, $\hat{\mu}_1 = \bar{X}_1$, $\hat{\mu}_2 = \bar{X}_2$, $\hat{\sigma}_1^2 = \sum_{j=1}^{n_1}\left(X_{1j} - \bar{X}_1\right)^2/n_1$ and $\hat{\sigma}_2^2 = \sum_{j=1}^{n_2}\left(X_{2j} - \bar{X}_2\right)^2/n_2$ are the ML estimators of μ_1, μ_2, σ_1^2 and σ_2^2 respectively.

3. Approximations for Weitzman coefficient Δ

In this section, we give three approximations for the Weitzman coefficient Δ based on some integral approximation rules and in the next section we will give the estimators of these approximations based on the two independent random samples $(X_{11,}X_{12}, \ldots . X_{1n_1})$ and $(X_{21,}X_{22}, \ldots . X_{2n_2})$.

Define the function $q(x) = min\{f_1(x, \mu_1, \sigma_1^2), f_2(x, \mu_2, \sigma_2^2)\}$, where $f_i(x; \mu_i, \sigma_i^2), i = 1, 2$ are defined in the previous section. The Weitzman coefficient Δ can be written as the following,

$$\Delta = \int_{-\infty}^{\infty} q(x)dx$$

Let $g(x)$ be any continuous increasing function in x and consider the transformation $u = g(x)$, then $x = g^{-1}(u) = (w(u), say)$ and $dx = w'(u)du$. Therefore, Δ can be expressed as follow,

$$\Delta = \int q(w(u))w'(u)du$$

In particular, we interest the case where $g(x)$ is a continuous cumulative distribution function, so that $\lim_{x\to-\infty} g(x) = 0$ and $\lim_{x\to\infty} g(x) = 1$. Therefore,

$$\Delta = \int_0^1 q(w(u))w'(u)du\,.$$

For more simplicity, let

$$\begin{aligned} h(u) &= q(w(u))w'(u) \\ &= min\{f_1(w(u), \mu_1, \sigma_1^2), f_2(w(u), \mu_2, \sigma_2^2)\}w'(u) \end{aligned}$$

then,

$$\Delta = \int_0^1 h(u)du\,.$$

To approximate the last integral, the interval $(0, 1)$ is divided into k subintervals each of length $1/k$. Let $u_i = i/k,\ i = 0, 1, 2, \ldots, k$ then the bounds of the k subintervals are as follows,

$$0 = u_0 < u_1 < u_2 < \ldots < u_k = 1\,,$$

Now, $\Delta = \int_0^1 h(u)du$ can be approximated by using the following three interested numerical integral rules known as, trapezoidal, Simpson 1/3 and Simpson 3/8 rules (Atkinson, 1989),

- The approximation of Δ by using trapezoidal rule (denoted by Δ_1) is,

$$\Delta_1 \cong \frac{1}{2k}\left[\lim_{u\to 0} h(u) + 2\sum_{j=1}^{k-1} h(u_j) + \lim_{u\to 1} h(u)\right]$$

- The approximation of Δ by using Simpson 1/3 rule (denoted by Δ_2) is,

$$\Delta_2 \cong \frac{1}{3k}\left[\lim_{u\to 0} h(u) + 4\sum_{j=1}^{k/2} h(u_{2j-1}) + 2\sum_{j=1}^{k/2-1} h(u_{2j}) + \lim_{u\to 1} h(u)\right]$$

- The approximation of Δ by using Simpson 3/8 rule (denoted by Δ_3) is,

$$\Delta_3 \cong \frac{3}{8K}\left\{\lim_{u\to 0} h(u) + 3\sum_{\substack{j=1\\ j\neq 3m}}^{k-1} h(u_j) + 2\sum_{j=1}^{\frac{k}{3}-1} h(u_{3j}) + \lim_{u\to 1} h(u)\right\},\ m \in N_0.$$

The above approximation expressions of Δ are still depending on unknown quantities. To use the above three expressions in practice, two quantities are to be determined. The first one is the transformation $g(x)$ and hence $w(u)$. The second quantity is the number of partitions k. In this paper, our special interest is to take g to be any continuous cumulative distribution function with support $(-\infty, \infty)$. Let T be a continuous random variable with cumulative distribution function $G_T(t)$ given by,

$$G_T(x) = 1 - \frac{1}{(1+e^x)^{\alpha}},\ -\infty < x < \infty,\ \alpha > 0.$$

That is, T has a generalized Logistic distribution with pdf,

$$g_T(x) = \frac{\alpha e^{-\alpha x}}{(1+e^x)^{\alpha+1}},\ -\infty < x < \infty,\ \alpha > 0.$$

In this case, $u = 1 - (1+e^x)^{-\alpha}$ with inverse transformation $x = w(u) = ln\left((1-u)^{-\frac{1}{\alpha}} - 1\right)$ and $w'(u) = \frac{1}{\alpha(1-u)\left(1-(1-u)^{1/\alpha}\right)}$.

The parameter α in the above transformation is under user control. Mathematically, any selection for $\alpha > 0$ is possible. However, to examine its practical effect on the performance of the proposed estimators, the two values $\alpha = 1.0$ and $= 0.5$ were taken into account in our simulation study in the next section.

The second quantity to be determined is k. A preliminary simulation study indicates that the selection of $k = \min\{n_1, n_2\}$ is very satisfactory, which was used in the simulation study in the next section.

4. Estimation of Δ

Let $f_i(x; \hat{\mu}_i, \hat{\sigma}_i^2) = (2\pi\hat{\sigma}_i^2)^{-1/2} e^{-(x-\hat{\mu}_i)^2/2\hat{\sigma}_i^2}$ be the ML estimator of $f_i(x; \mu_i, \sigma_i^2) = (2\pi\sigma_i^2)^{-1/2} e^{-(x-\mu_i)^2/2\sigma_i^2}$, $i = 1, 2$. Also, let $\hat{h}(u) = min\{\hat{f}_1(w(u); \hat{\mu}_1, \hat{\sigma}_1^2), \hat{f}_2(w(u); \hat{\mu}_2, \hat{\sigma}_2^2)\}w'(u)$ be the ML estimator of $h(u)$. If the transformation $g_T(x)$ is considered to be the generalized Logistic distribution as described in the previous section then $\lim_{u\to 0} h(u) = \lim_{u\to 1} h(u) = 0$. Therefore, the proposed estimators of Δ are as shown below,

- The proposed estimator of Δ based on Trapezoidal rule is,

$$\hat{\Delta}_{Trap} = \frac{1}{k}\sum_{j=1}^{k-1} \hat{h}(j/k) .$$

- The proposed estimator of Δ based on Simpson 1/3 rule is,

$$\hat{\Delta}_{Sim1} = \frac{1}{3k}\left[4\sum_{j=1}^{k/2} \hat{h}((2j-1)/k) + 2\sum_{j=1}^{k/2-1} \hat{h}(2j/k)\right].$$

- The proposed estimator of Δ based on Simpson 3/8 rule is,

$$\hat{\Delta}_{Sim2} = \frac{3}{8k}\left\{3\sum_{\substack{j=1\\ j\neq 3m}}^{k-1} \hat{h}(j/k) + 2\sum_{j=1}^{k/3-1} \hat{h}(3j/k)\right\}, \ m \in N_0$$

5. Simulation study and results

In this simulation study, the four estimators $\hat{\Delta}_{ED}$, $\hat{\Delta}_{Trap}$, $\hat{\Delta}_{Simp1}$ and $\hat{\Delta}_{Simp2}$ of Δ are considered. The estimator $\hat{\Delta}_{ED}$ that developed by Eidous and Al-Daradkeh (2023) (see Section 2) is considered in this study. The other estimators in Section (2) cannot be considered unless the required assumptions are valid. Therefore, we only studied the $\hat{\Delta}_{ED}$ estimator for the purpose of comparison with the proposed estimators.

Assume that we have two independent random samples $x_{11}, x_{12}, ..., x_{1n_1}$ of size n_1 and $x_{21}, x_{22}, ..., x_{2n_2}$ of size n_2, where the first sample is generated from $N(\mu_1 = 0,\ \sigma_1^2 = 1)$ and the second sample is simulated from $N(\mu_2, \sigma_2^2)$, where $(\mu_2, \sigma_2^2) = (-0.2, 1.1),\ (2.5, 4),\ (3.5, 1.5),\ (10, 2.5)$. These values of μ_1, σ_1^2 μ_2, σ_2^2 were chosen to vary the exact values of Δ between 0 and 1.

From each pair of distributions, 1000 samples of sizes $(n_1, n_2) = (24,30)$, $(54,54)$, $(96,180)$ were simulated independently from the two normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ with selected parameters as given above (see also, Table 1). The values of the sample sizes were selected so that they are divisible by 2 and 3 so that we can calculate the estimators $\widehat{\Delta}_{Simp1}$ and $\widehat{\Delta}_{Simp2}$ respectively and the number of partitions k is taken to be $k = \min\{n_1, n_2\}$.

The empirical results are given in Table (1). For each estimator, we computed the relative bias (RB), relative root mean square error (RRMSE) and efficiency (EFF). These measures are defined as follows:

Let $\hat{\theta}$ be a specific estimator for a parameter θ (exact value), and let $\hat{\theta}_{(j)}$ be the observed value of $\hat{\theta}$ based on iteration j, $j = 1, 2, \ldots, R = 1000$, then,

$$RB = \frac{\widehat{E}(\hat{\theta}) - \theta}{\theta},$$

$$RRMSE = \frac{\sqrt{\widehat{MSE}(\hat{\theta})}}{\theta}$$

and the efficiency of the proposed estimator (Prop) with respect to Eidous and Al-Daradkeh (2022) estimator ($\widehat{\Delta}_{ED}$) is defined by,

$$EFF = \frac{\widehat{MSE}(\widehat{\Delta}_{ED})}{\widehat{MSE}\,(Prop)},$$

where $\widehat{E}(\hat{\theta}) = \frac{\sum_{j=1}^{R} \hat{\theta}_{(j)}}{R}$ and $\widehat{MSE}(\hat{\theta}) = \frac{\sum_{j=1}^{R} \left(\hat{\theta}_{(j)} - \theta\right)^2}{R}$.

All simulation results are calculated by using Mathematica, Version 11. Based on these, which presented in Table (1), the general conclusions are:

1. It is clear that the |RBs| and RMSEs of the different estimators decrease as the sample sizes increases. This result is very clear if we compare the values of |RB|s and RRMSE for the different estimators when $(n_1, n_2) = (24,30)$ with their values when $(n_1, n_2) = (96, 180)$. This indicates that the various estimators are consistent estimators.

2. The proposed estimators are more efficient than the estimator suggested by Eidous and Al-Daradkeh (2023) in most considered cases. This can be depicted if one examines the corresponding values of the RRMSE and EFF especially when the exact values of Δ become small toward 0.

3. By comparing the performances of the proposed estimators of Δ together, it is clear that their performances are close to each other. Also, their performance coincided for large sample sizes. This indicates that the three numerical integration rules; trapezoidal, Simpson 1/3, and Simpson 3/8 rules give the same results in estimating Δ.

4. By examining and comparing the results corresponding to the two transformations $1-(1+e^x)^{-1}$ (i.e. $\alpha = 1.0$) and $1-(1+e^x)^{-1/2}$ (i.e. $\alpha = 1/2$), we find that there is no better transformation than the other for all considered cases. Although, there is a preference for $\alpha = 1.0$ over $\alpha = 1/2$ in some cases, but the converse occurs in other cases. However, it appears that the proposed estimators are sensitive to how the suitable transformation is chosen and one should take a care when deciding to choose the transformation. At least and depending on the simulation results, the above two transformations work well in estimating the OVL coefficient Δ.

Table (1). The RB, RRMSE and EFF of the estimators $\widehat{\Delta}_{ED}$, $\widehat{\Delta}_{Trap}$, $\widehat{\Delta}_{sim1}$ and $\widehat{\Delta}_{sim2}$ when the data are simulated from pair normal distributions a) $N(0,1)$ and $N(-0.2,1.1)$ b) $N(0,1)$ and $N(2.5,4)$ c) $N(0,1)$ and $N(3.5,1.5)$ d) $N(0,1)$ and $N(10,2.5)$ and e) $N(0,1)$ and $N(10,2.5)$.

a) The exact $\Delta = 0.9151$

		α = 1				α = 1/2			
(n_1, n_2)		$\widehat{\Delta}_{ED}$	$\widehat{\Delta}_{Trap}$	$\widehat{\Delta}_{Sim1}$	$\widehat{\Delta}_{Sim2}$	$\widehat{\Delta}_{Dar}$	$\widehat{\Delta}_{Trap}$	$\widehat{\Delta}_{Sim1}$	$\widehat{\Delta}_{Sim2}$
(24, 30)	RB	-0.074	-0.074	-0.074	-0.074	-0.066	-0.067	-0.067	-0.066
	RRMSE	0.114	0.113	0.113	0.113	0.107	0.107	0.107	0.107
	EFF	**1.000**	**1.015**	**1.014**	**1.015**	**1.000**	**0.999**	**0.997**	**1.001**
(54, 54)	RB	-0.034	-0.034	-0.034	-0.034	-0.035	-0.035	-0.035	-0.035
	RRMSE	0.071	0.071	0.071	0.071	0.073	0.073	0.073	0.073
	EFF	**1.000**	**1.005**	**1.005**	**1.005**	**1.000**	**1.000**	**1.000**	**0.999**
(96, 180)	RB	-0.014	-0.014	-0.014	-0.014	-0.012	-0.012	-0.012	-0.012
	RRMSE	0.049	0.048	0.048	0.048	0.048	0.047	0.047	0.047
	EFF	**1.000**	**1.003**	**1.003**	**1.003**	**1.000**	**1.006**	**1.006**	**1.006**

b) The exact $\Delta = 0.6099$

		α = 1				α = 1/2			
(n_1, n_2)		$\widehat{\Delta}_{ED}$	$\widehat{\Delta}_{Trap}$	$\widehat{\Delta}_{Sim1}$	$\widehat{\Delta}_{Sim2}$	$\widehat{\Delta}_{Dar}$	$\widehat{\Delta}_{Trap}$	$\widehat{\Delta}_{Sim1}$	$\widehat{\Delta}_{Sim2}$
(24, 30)	RB	-0.020	-0.022	-0.022	-0.022	-0.022	-0.025	-0.022	-0.021

	RRMSE	0.142	0.136	0.136	0.137	0.145	0.138	0.139	0.139
	EFF	**1.000**	**1.093**	**1.083**	**1.079**	**1.000**	**1.095**	**1.078**	**1.084**
(54, 54)	RB	-0.012	-0.010	-0.010	-0.012	-0.012	-0.012	-0.012	-0.012
	RRMSE	0.100	0.095	0.095	0.095	0.101	0.096	0.096	0.096
	EFF	**1.000**	**1.114**	**1.114**	**1.113**	**1.000**	**1.113**	**1.110**	**1.110**
(96, 180)	RB	-0.004	-0.004	-0.004	-0.004	-0.001	-0.001	-0.001	-0.001
	RRMSE	0.064	0.061	0.061	0.061	0.064	0.062	0.062	0.062
	EFF	**1.000**	**1.101**	**1.102**	**1.102**	**1.000**	**1.054**	**1.055**	**1.054**

c) The exact $\Delta = 0.3577$

		$\alpha = 1$				$\alpha = 1/2$			
(n_1, n_2)		$\widehat{\Delta}_{ED}$	$\widehat{\Delta}_{Trap}$	$\widehat{\Delta}_{Sim1}$	$\widehat{\Delta}_{Sim2}$	$\widehat{\Delta}_{Dar}$	$\widehat{\Delta}_{Trap}$	$\widehat{\Delta}_{Sim1}$	$\widehat{\Delta}_{Sim2}$
(24, 30)	RB	-0.033	-0.028	-0.027	-0.027	-0.021	-0.019	-0.014	-0.011
	RRMSE	0.186	0.158	0.159	0.159	0.195	0.161	0.166	0.164
	EFF	**1.000**	**1.384**	**1.364**	**1.363**	**1.000**	**1.464**	**1.391**	**1.425**
(54, 54)	RB	-0.006	-0.007	-0.007	-0.007	-0.010	-0.011	-0.011	-0.012
	RRMSE	0.1398	0.114	0.114	0.114	0.128	0.107	0.107	0.107
	EFF	**1.000**	**1.485**	**1.485**	**1.484**	**1.000**	**1.429**	**1.423**	**1.422**
(96, 180)	RB	-0.009	-0.008	-0.008	-0.008	-0.005	-0.005	-0.005	-0.005
	RRMSE	0.085	0.071	0.071	0.071	0.081	0.070	0.070	0.070
	EFF	**1.000**	**1.416**	**1.417**	**1.417**	**1.000**	**1.336**	**1.335**	**1.334**

d) The exact $\Delta = 0.1573$

		$\alpha = 1$				$\alpha = 1/2$			
(n_1, n_2)		$\widehat{\Delta}_{ED}$	$\widehat{\Delta}_{Trap}$	$\widehat{\Delta}_{Sim1}$	$\widehat{\Delta}_{Sim2}$	$\widehat{\Delta}_{Dar}$	$\widehat{\Delta}_{Trap}$	$\widehat{\Delta}_{Sim1}$	$\widehat{\Delta}_{Sim2}$
(24, 30)	RB	-0.013	-0.006	-0.007	-0.004	-0.013	-0.006	-0.006	-0.006
	RRMSE	0.372	0.349	0.350	0.350	0.360	0.340	0.340	0.340
	EFF	**1.000**	**1.137**	**1.131**	**1.129**	**1.000**	**1.119**	**1.120**	**1.120**
(54, 54)	RB	0.001	0.003	0.003	0.003	-0.021	-0.018	-0.018	-0.018
	RRMSE	0.270	0.258	0.258	0.258	0.259	0.246	0.246	0.246
	EFF	**1.000**	**1.088**	**1.088**	**1.088**	**1.000**	**1.109**	**1.109**	**1.109**

(96, 180)	RB	-0.005	0	0	0	-0.003	0	0	0
	RRMSE	0.173	0.166	0.166	0.166	0.156	0.149	0.149	0.149
	EFF	**1.000**	**1.090**	**1.089**	**1.090**	**1.000**	**1.095**	**1.094**	**1.095**

e) The exact $\Delta = 0.0039$.

		$\alpha = 1$				$\alpha = 1/2$			
(n_1, n_2)		$\widehat{\Delta}_{ED}$	$\widehat{\Delta}_{Trap}$	$\widehat{\Delta}_{Sim1}$	$\widehat{\Delta}_{Sim2}$	$\widehat{\Delta}_{Dar}$	$\widehat{\Delta}_{Trap}$	$\widehat{\Delta}_{Sim1}$	$\widehat{\Delta}_{Sim2}$
(24, 30)	RB	-0.002	0.274	0.410	0.364	-0.175	0.308	0.305	0.312
	RRMSE	1.808	1.268	1.409	1.360	1.625	1.359	1.357	1.364
	EFF	**1.000**	**2.033**	**1.644**	**1.765**	**1.000**	**1.430**	**1.434**	**1.419**
(54, 54)	RB	-0.054	0.157	0.145	0.157	-0.083	0.119	0.119	0.119
	RRMSE	1.310	0.866	0.863	0.862	1.252	0.795	0.796	0.796
	EFF	**1.000**	**2.284**	**2.302**	**2.308**	**1.000**	**2.477**	**2.473**	**2.474**
(96, 180)	RB	-0.025	0.071	0.069	0.072	-0.011	0.057	0.057	0.057
	RRMSE	0.783	0.470	0.469	0.470	0.812	0.448	0.448	0.448
	EFF	**1.000**	**2.774**	**2.782**	**2.768**	**1.000**	**3.278**	**3.280**	**3.279**

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