SCIREA Journal of Electrics, Communication and Automatic Control

http://www.scirea.org/journal/ECAC

October 15, 2016

Volume 1, Issue1, October 2016

Asymptotic statistical analysis of virtual reference feedback tuning control

Hong Wang-jian[1], Guo Xiao-yong[2]

[1]Dipartimento di Elettronica, Informazione Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy

[2]School of Science, Henan University of Engineering, Zhengzhou 451191, China

Abstract:

Virtual reference feedback tuning control is a data-driven control strategy. No model identification of the plant is needed in this method. As the asymptotic covariance matrix is an important factor in the whole system identification theory. So here the error about the unknown parameter estimation is derived through Taylor series expression. Then the corresponding covariance matrix of the parameter estimation error is established. The two diagonal sub-matrices in the covariance matrix are obtained using some matrix operations. These two diagonal sub- matrices are the asymptotic covariance matrix expression of the two unknown parameter estimation vectors in the closed-loop system. Based on this asymptotic covariance matrix, an optimal filter is obtained by solving an optimization problem which includes some trace operation. Finally, the efficiency and possibility of the proposed strategy can be confirmed by the simulation example results.

Keywords: Virtual reference feedback tuning control; Asymptotic analysis; Stochastic optimization

1. Introduction

Virtual reference feedback tuning control is one data driven control method. When the controller is parameterized by one unknown parameter vector, virtual reference feedback tuning control tunes the unknown parameter vector in the controller by using only input-output measured data without any knowledge of plant model. In the idea of virtual reference feedback tuning control, the process of designing controller is transformed into tuning the unknown parameter vector.

The idea of virtual reference feedback tuning control was first proposed in reference [1], in reference [2] the author generalized this idea completely to design one degree of freedom controller. Then the idea of virtual reference feedback tuning control was applied in a benchmark problem [3]. When consider two degrees of freedom controllers, reference [4] summarized that the problem of designing controllers was changed to a problem of identification parameter. One iterative optimization algorithm was introduced to obtain the unknown parameter vector in nonlinear controller by using virtual reference feedback tuning [5]. To decrease the computational complexity, some improvements about the classical gradient algorithm were used in virtual reference feedback tuning [6]. In reference [7], one improved virtual reference feedback tuning control-iterative feedback tuning method was given through two groups of data points.

Because the main essence of virtual reference feedback tuning control is that it transfers the problem of designing controller into an identification problem, then many classical optimization algorithms can be introduced to identify these controller parameters. From above references, we see that many research about this virtual reference feedback tuning are concentrated on the identification algorithm and its application, it means that all references study the problem of how to identify the unknown parameter vector in the parameterized controller. So there is no any research on the input signal design or model validation theory about this virtual reference feedback tuning control. When studying the problem of input signal design, the common way is to solve one optimization problem with respect to the input power spectral, and the cost function in that optimization problem depends on one matrix norm of the asymptotic variance matrix about the unknown parameter vector in the controller [8]. Furthermore in the model validation process, the controller parameter estimations may be included in one guaranteed ellipsoid [9]. Similarly the computation of that guaranteed ellipsoid needs the asymptotic variance matrix about the unknown parameter vector in the controller. So from these two subjects, we think that this asymptotic variance matrix of the unknown parameter vector is one important index to measure the identification accuracy and it will be used for the latter two subjects-optimal input design and model validation. Based on above descriptions, in this short note we study the asymptotic statistical analysis of virtual reference tuning control and derive the asymptotic variance matrix of the unknown parameter vector to obtain one optimal filter form.

2. Virtual reference feedback tuning control

Assume the plant model is denoted by one rational transfer function $P(z)$, and $P(z)$ is unknown. Then the whole input-output relation is described as

$$y(t) = P(z)u(t) + v(t) \tag{1}$$

Where z is a time shift operator ($zu(t) = u(t-1)$), $u(t)$ is the measured input, $y(t)$ is the measured output corresponding to the plant model $P(z)$, and $v(t)$ is the external noise. Furthermore assume noise $v(t)$ is a quasi-stationary stochastic signal and express it as.

$$v(t) = H(z)e(t)$$

Where $e(t)$ is zero mean white noise with variance σ_e^2, $H(z)$ is a shaping filter which is used to filter the white noise $e(t)$. Now consider the following closed loop system structure with two degrees of freedom in Fig.1.

$$\begin{cases} y(t) = P(z)u(t) + v(t) \\ u(t) = C_1(z,\theta)\left[r(t) - C_2(z,\eta)y(t)\right] \end{cases} \tag{2}$$

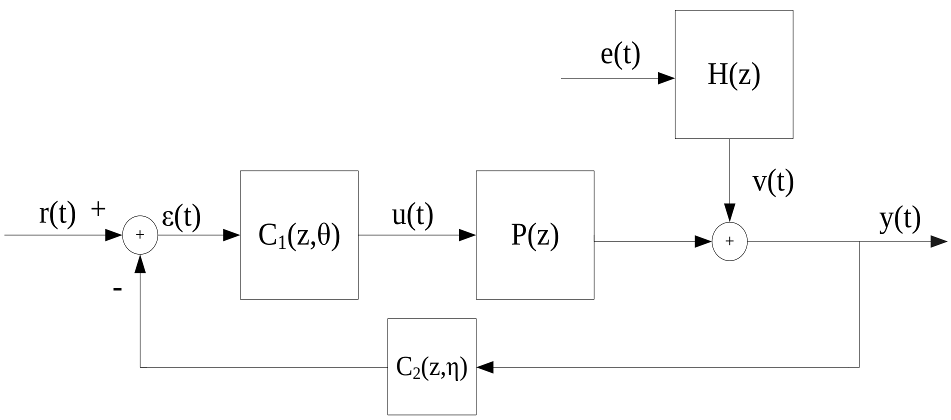

Fig.1: The closed loop system with two degrees of freedom

In equation (2) $r(t)$ is the excited signal, $(C_1(z,\theta),\ C_2(z,\eta))$ are two controllers. These two controllers are parameterized as linear forms.

$$
\begin{aligned}
&C_1(z,\theta)=\alpha^T(z)\theta,\quad C_2(z,\eta)=\beta^T(z)\eta \\
&\alpha(z)=\left[\alpha_1(z) \quad \cdots \quad \alpha_n(z)\right]^T, \beta(z)=\left[\beta_1(z) \quad \cdots \quad \beta_n(z)\right]^T \\
&\theta=\left[\theta_1 \quad \theta_2 \quad \cdots \quad \theta_n\right]^T, \eta=\left[\eta_1 \quad \eta_2 \quad \cdots \quad \eta_n\right]^T
\end{aligned}
\tag{3}
$$

Where $\alpha(z)$ and $\beta(z)$ denote two known vector of linear discrete time transfer functions, θ and η are two n dimensional unknown vectors of parameters.

Analyzing (1), we get one transfer function of the closed loop system.

$$
y(t)=\frac{P(z)C_1(z,\theta)r(t)+v(t)}{1+P(z)C_1(z,\theta)C_2(z,\eta)} \tag{4}
$$

Assume one expected model $M(z)$ is given in the model reference adaptive control structure. To avoid the identification process about $P(z)$, virtual reference feedback control is applied to directly solve the two unknown parameter vectors in the controllers.

Collect a set of input $u(t)$ and output $y(t)$ about the plant model $P(z)$ through open loop or closed loop experiment, define one virtual reference signal $\bar{r}(t)$ such that:

$$
y(t)=M(z)\bar{r}(t) \tag{5}
$$

The virtual reference signal means that we can obtain the input signal by using the following relation.

$$
\bar{r}(t)=M^{-1}(z)y(t)
$$

Applying this virtual reference signal $\bar{r}(t)$ as the excitation signal in the closed loop system, we get the corresponding output data $y(t)$. Define the reference tracking error as:

$$
\varepsilon(t)=\bar{r}(t)-C_2(z,\eta)y(t) \tag{6}
$$

Applying this reference tracking error to the controller $C_1(z,\theta)$ in one virtual experiment, the corresponding output signal is get.

$$
u(t)=C_1(z,\theta)\varepsilon(t) \tag{7}
$$

Then the input and output data about controller $C_1(z,\theta)$ are the tracking error $\varepsilon(t)$ and signal $u(t)$ respectively. Using these data, two parameter vectors are obtained by solving the following optimization problem.

$$\min_{\theta,\eta} J_{VR}^N(\theta,\eta)=\frac{1}{N}\sum_{t=1}^{N}\left[u(t)-C_1(z,\theta)\varepsilon(t)\right]^2=\frac{1}{N}\sum_{t=1}^{N}\left[u(t)-C_1(z,\theta)\left[M^{-1}(z)-C_2(z,\eta)\right]y(t)\right]^2 \tag{8}$$

Consider the objective function of the virtual reference approach (8), all variables are known but the two controllers $C_1(z,\theta)$ and $C_2(z,\eta)$ and the plant model $P(z)$ does not exist in the objective function.

Pre-filtering the signal $\varepsilon(t)$ and $u(t)$ with a suitable filter $L(z)$, we obtain the filter signals

$$\varepsilon_L(t)=L(z)\varepsilon(t),\quad u_L(t)=L(z)u(t), y_L(t)=L(z)y(t) \tag{9}$$

After introducing filter $L(z)$, equation (8) is re-written as.

$$J_{VR}^N(\theta,\eta)=\frac{1}{N}\sum_{t=1}^{N}\left[u_L(t)-C_1(\theta)\left[M^{-1}(z)-C_2(\eta)\right]y_L(t)\right]^2 \tag{10}$$

Reformulating equation (10) again as.

$$J_{VR}^N(\theta,\eta)=\frac{1}{N}\sum_{t=1}^{N}\left[u_L(t)-C_1(\theta)\varepsilon_L(t)\right]^2=\frac{1}{N}\sum_{t=1}^{N}\left[Mu_L(t)-C_1(\theta)y_L(t)+MC_1(\theta)C_2(\eta)y_L(t)\right]^2 \tag{11}$$

3. Asymptotic statistical analysis of two unknown parameter vectors

Applying the data set $Z^N=\{y(t),u(t)\}_{t=1}^N$, where N is the number of data points, the unknown parameter vector estimations are identified by solving the following optimization problem.

$$\left(\hat{\theta}_N,\hat{\eta}_N\right)=\arg\min_{(\theta,\eta)} J_{VR}^N(\theta,\eta)$$

The asymptotic estimations corresponding to the unknown parameter vector are defined as the following limit.

$$\left(\theta^*,\eta^*\right)=\arg\min_{(\theta,\eta)}\lim_{N\to\infty} E\left\{J_{VR}^N(\theta,\eta)\right\}$$

Where E is the expectation operation, and $E\left\{J_{VR}^N(\theta,\eta)\right\}$ is denoted as.

$$E\left\{J_{VR}^N(\theta,\eta)\right\}=E\left[Mu_L(t)-C_1(\theta)y_L(t)+MC_1(\theta)C_2(\eta)y_L(t)\right]^2$$

In the probability sense, the following asymptotic relations hold.

$$J_{VR}^N(\theta,\eta)\to J_{VR}(\theta,\eta)=E\left\{J_{VR}^N(\theta,\eta)\right\},$$
$$\left(\hat{\theta}_N,\hat{\eta}_N\right)\to\left(\theta^*,\eta^*\right), N\to\infty$$

Fromm the first order optimal necessary condition, we see that if $\left(\hat{\theta}_N,\hat{\eta}_N\right)$ and $\left(\theta^*,\eta^*\right)$ are minimum values with respect to their cost functions, the following relations hold.

$$\begin{cases}\left[\dfrac{\partial J_{VR}^N(\theta,\eta)}{\partial\theta}\right]_{\left(\hat{\theta}_N,\hat{\eta}_N\right)}=0\\ \left[\dfrac{\partial J_{VR}^N(\theta,\eta)}{\partial\eta}\right]_{\left(\hat{\theta}_N,\hat{\eta}_N\right)}=0\end{cases},\quad \begin{cases}\left[\dfrac{\partial J_{VR}(\theta,\eta)}{\partial\theta}\right]_{\left(\theta^*,\eta^*\right)}=0\\ \left[\dfrac{\partial J_{VR}(\theta,\eta)}{\partial\eta}\right]_{\left(\theta^*,\eta^*\right)}=0\end{cases}$$

Expanding above equation, we have.

$$\begin{aligned}&\frac{2}{N}\sum_{t=1}^{N}\left[Mu_L-C_1(\theta)y_L+MC_1(\theta)C_2(\eta)y_L\right]\left[-\frac{\partial C_1(\theta)}{\partial\theta}y_L+M\frac{\partial C_1(\theta)}{\partial\theta}C_2(\eta)y_L\right]=0\\&\frac{2}{N}\sum_{t=1}^{N}\left[Mu_L-C_1(\theta)y_L+MC_1(\theta)C_2(\eta)y_L\right]\left[MC_1(\theta)\frac{\partial C_2(\eta)}{\partial\eta}y_L\right]=0\end{aligned}\tag{12}$$

Let $\xi=(\theta,\eta)$, then equation (12) can be re-written as.

$$\begin{aligned}\hat{\xi}_N&=\left(\hat{\theta}_N,\hat{\eta}_N\right)=\arg\min_{\xi=(\theta,\eta)}J_{VR}^N(\xi)=\arg\min_{\xi=(\theta,\eta)}J_{VR}^N(\theta,\eta)\\ \xi^*&=\left(\theta^*,\eta^*\right)=\arg\min_{\xi=(\theta,\eta)}J_{VR}(\xi)=\arg\min_{\xi=(\theta,\eta)}J_{VR}(\theta,\eta)\\ &\left[\frac{\partial J_{VR}^N(\xi)}{\partial\xi}\right]_{\hat{\xi}_N}=\begin{bmatrix}\dfrac{\partial J_{VR}^N(\xi)}{\partial\theta}\\ \dfrac{\partial J_{VR}^N(\xi)}{\partial\eta}\end{bmatrix}_{\hat{\xi}_N}=0\end{aligned}$$

Using Taylor series expansion with two order form, the first-order partial derivative can be expanded around the asymptotic limit point ξ^*.

$$\left[\frac{\partial J_{VR}^N(\xi)}{\partial\xi}\right]_{\hat{\xi}_N}=\left[\frac{\partial J_{VR}^N(\xi)}{\partial\xi}\right]_{\xi^*}+\left[\frac{\partial^2 J_{VR}^N(\xi)}{\partial\xi^2}\right]_{\xi^*}\left(\hat{\xi}_N-\xi^*\right)\tag{13}$$

Expanding equation (13), we obtain.

$$\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}\dfrac{\partial J_{VR}^N(\xi)}{\partial\theta}\\ \dfrac{\partial J_{VR}^N(\xi)}{\partial\eta}\end{bmatrix}_{\xi^*}+\begin{bmatrix}\dfrac{\partial^2 J_{VR}^N(\xi)}{\partial\theta^2}\\ \dfrac{\partial^2 J_{VR}^N(\xi)}{\partial\eta^2}\end{bmatrix}_{\xi^*}\begin{bmatrix}\hat{\theta}_N-\theta^*\\ \hat{\eta}_N-\eta^*\end{bmatrix}\tag{14}$$

Again from equation (13), we see.

$$\left(\hat{\xi}_N - \xi^*\right) = -\begin{bmatrix} \frac{\partial^2 J_{VR}^N(\xi)}{\partial\theta^2} & \frac{\partial^2 J_{VR}^N(\xi)}{\partial\theta\partial\eta} \\ \frac{\partial^2 J_{VR}^N(\xi)}{\partial\eta\partial\theta} & \frac{\partial^2 J_{VR}^N(\xi)}{\partial\eta^2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial J_{VR}^N(\xi)}{\partial\theta} \\ \frac{\partial J_{VR}^N(\xi)}{\partial\eta} \end{bmatrix} \tag{15}$$

From equation (15), the asymptotic result about the unknown parameter vector ξ is derived as.

$$\sqrt{N}\left(\hat{\xi}_N - \xi^*\right) \to \square\,(0, A), as\ N \to \infty$$

It means that the unknown parameter vector estimation $\hat{\xi}_N$ will converge to its true asymptotic limit value ξ^*, and the estimation error converges to a Gaussian stochastic variable. Its asymptotic variance matrix is given as follows.

$$A = \begin{bmatrix} A_\theta & A_{\theta\eta} \\ A_{\eta\theta} & A_\eta \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 J_{VR}^N(\xi)}{\partial\theta^2} & \frac{\partial^2 J_{VR}^N(\xi)}{\partial\theta\partial\eta} \\ \frac{\partial^2 J_{VR}^N(\xi)}{\partial\eta\partial\theta} & \frac{\partial^2 J_{VR}^N(\xi)}{\partial\eta^2} \end{bmatrix}^{-1} \operatorname{cov}\begin{bmatrix} \frac{\partial J_{VR}^N(\xi)}{\partial\theta} \\ \frac{\partial J_{VR}^N(\xi)}{\partial\eta} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 J_{VR}^N(\xi)}{\partial\theta^2} & \frac{\partial^2 J_{VR}^N(\xi)}{\partial\theta\partial\eta} \\ \frac{\partial^2 J_{VR}^N(\xi)}{\partial\eta\partial\theta} & \frac{\partial^2 J_{VR}^N(\xi)}{\partial\eta^2} \end{bmatrix}^{-T} \tag{16}$$

Equation (16) means that these two parameter estimations are two stochastic variables. But it is very difficult to compute the asymptotic variance matrix A. Now we start to derive the main diagonal elements in matrix A, as these main diagonal elements are the asymptotic variance sub-matrices with respect to these two unknown parameter vectors (θ, η).

Through basic partial derivative calculation, we obtain.

$$\begin{aligned} \frac{\partial J_{VR}^N(\xi)}{\partial\theta} &= \frac{2}{N}\sum_{t=1}^{N}\left[Mu_L - C_1(\theta)y_L + MC_1(\theta)C_2(\eta)y_L\right]\left[-\frac{\partial C_1(\theta)}{\partial\theta}y_L + M\frac{\partial C_1(\theta)}{\partial\theta}C_2(\eta)y_L\right] \\ \frac{\partial J_{VR}^N(\xi)}{\partial\eta} &= \frac{2}{N}\sum_{t=1}^{N}\left[Mu_L - C_1(\theta)y_L + MC_1(\theta)C_2(\eta)y_L\right]\left[MC_1(\theta)\frac{\partial C_2(\eta)}{\partial\eta}y_L\right] \end{aligned} \tag{17}$$

Using the parameterized controllers, we get.

$$\frac{\partial^2 C_1(\theta)}{\partial\theta^2} = \frac{\partial^2 C_2(\eta)}{\partial\eta^2} = 0$$

Through basic partial derivative calculation again, we obtain.

$$\frac{\partial^2 J_{VR}^N(\xi)}{\partial\theta^2} = \frac{2}{N}\sum_{t=1}^{N}\left[-\frac{\partial C_1(\theta)}{\partial\theta}y_L + M\frac{\partial C_1(\theta)}{\partial\theta}C_2(\eta)y_L\right]^2$$

$$\begin{aligned}\frac{\partial^2 J_{VR}^N(\xi)}{\partial\theta\partial\eta} &= \frac{2}{N}\sum_{t=1}^{N}\left[MC_1(\theta)\frac{\partial C_2(\eta)}{\partial\eta}y_L\right]\left[-\frac{\partial C_1(\theta)}{\partial\theta}y_L + M\frac{\partial C_1(\theta)}{\partial\theta}C_2(\eta)y_L\right] \\ &+ \frac{2}{N}\sum_{t=1}^{N}\left[Mu_L - C_1(\theta)y_L + MC_1(\theta)C_2(\eta)y_L\right]\left[M\frac{\partial C_1(\theta)}{\partial\theta}\frac{\partial C_2(\eta)}{\partial\eta}y_L\right]\end{aligned} \tag{18}$$

$$\frac{\partial^2 J_{VR}^N(\xi)}{\partial\eta^2} = \frac{2}{N}\sum_{t=1}^{N}\left[MC_1(\theta)\frac{\partial C_2(\eta)}{\partial\eta}y_L\right]^2$$

$$\frac{\partial^2 J_{VR}^N(\xi)}{\partial\eta\partial\theta} = \frac{\partial^2 J_{VR}^N(\xi)}{\partial\theta\partial\eta}$$

Taking expectation operation with respect to equation (17), we derive that.

$$\begin{aligned}E\left[\frac{\partial J_{VR}^N(\xi)}{\partial\theta}\right] &= Mu_L\left[-C_1'(\theta) + MC_1'(\theta)C_2(\eta)\right]Pu_L \\ &+ \left[MC_1(\theta)C_2(\eta) - C_1(\theta)\right]\left[-C_1'(\theta) + MC_1'(\theta)C_2(\eta)\right]\left[P^2H^2 + H^2\sigma_e^2\right]\end{aligned} \tag{19}$$

Where in equation (19) we use the input-output relation (1), and apply another notation to express the derivative operation. Then the corresponding variance matrix is given as.

$$\operatorname{cov}\left[\frac{\partial J_{VR}^N(\xi)}{\partial\theta}\right] = E\left[\frac{\partial J_{VR}^N(\xi)}{\partial\theta} - E\left[\frac{\partial J_{VR}^N(\xi)}{\partial\theta}\right]\right]^2 = \left[\left(MC_2(\eta)-1\right)^2 C_1(\theta)C_1'(\theta)2PHu_Le\right]^2 \tag{20}$$

Using the classical Pasiphae theorem from digital signal processing theory, we re-written equation (20) as.

$$\operatorname{cov}\left[\frac{\partial J_{VR}^N(\xi)}{\partial\theta}\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(MC_2(\eta)-1\right)^4\left(C_1(\theta)C_1'(\theta)\right)^2 4P^2H^2\phi_{u_L}\sigma_e^2 \tag{21}$$

Let

$$x_1 = 4P^2\left(MC_2(\eta)-1\right)^4\left(C_1(\theta)C_1'(\theta)\right)^2$$

Then equation (21) can be simplified as.

$$\operatorname{cov}\left[\frac{\partial J_{VR}^N(\xi)}{\partial\theta}\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi}x_1H^2\phi_{u_L}\sigma_e^2 \tag{22}$$

Similarly we obtain other asymptotic variance sub-matrices.

$$\begin{aligned}&\operatorname{cov}\left[\frac{\partial J_{VR}^N(\xi)}{\partial\eta}\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi}x_2H^2\phi_{u_L}\sigma_e^2 \\ &x_2 = \left\{\left[2PC_1(\theta)\left[MC_2(\eta)-1\right] + M\right]MC_1(\theta)C_2'(\eta)\right\}^2\end{aligned} \tag{23}$$

The cross covariance estimation between two gradient estimations is that.

$$\operatorname{cov}\left(\frac{\partial J_{VR}^{N}(\xi)}{\partial\theta},\frac{\partial J_{VR}^{N}(\xi)}{\partial\eta}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}x_3 2PH^2\phi_{u_L}\sigma_e^2$$
$$x_3=\left(x_1x_2\right)^{\frac{1}{2}} \tag{24}$$

Substituting equation (22)、(23) and (24) into the variance matrix of first order gradient estimation, we obtain.

$$\operatorname{cov}\begin{bmatrix}\dfrac{\partial J_{VR}^{N}(\xi)}{\partial\theta}\\ \dfrac{\partial J_{VR}^{N}(\xi)}{\partial\eta}\end{bmatrix}=\begin{bmatrix}x_1H^2Eu_L^2\sigma_e^2 & x_3 2PH^2Eu_L^2\sigma_e^2\\ x_3 2PH^2Eu_L^2\sigma_e^2 & x_2H^2Eu_L^2\sigma_e^2\end{bmatrix} \tag{25}$$

When $N\to\infty$, the following asymptotic relations hold.

$$\left[\frac{\partial^2 J_{VR}^{N}(\xi)}{\partial\theta\partial\eta}\right]_{\hat{\xi}_N}\to E\left[\frac{\partial^2 J_{VR}^{N}(\xi)}{\partial\theta\partial\eta}\right]_{\xi^*}=x_4Eu_L^2+x_5\sigma_e^2$$
$$x_4=MC_1(\theta)C_2'(\eta)\left(-C_1'(\theta)+MC_1'(\theta)C_2(\eta)\right)H^2+\left(MC_2(\eta)-1\right)C_1(\theta)MC_1'(\theta)C_2'(\eta)P^2 \tag{26}$$
$$x_5=MC_1(\theta)C_2'(\eta)\left(-C_1'(\theta)+MC_1'(\theta)C_2(\eta)\right)H^2+\left(MC_2(\eta)-1\right)C_1(\theta)MC_1'(\theta)C_2'(\eta)H^2$$

$$\left[\frac{\partial^2 J_{VR}^{N}(\xi)}{\partial\eta^2}\right]_{\hat{\eta}_N}\to E\left[\frac{\partial^2 J_{VR}^{N}(\xi)}{\partial\eta^2}\right]_{\eta^*}=x_6Eu_L^2+x_7\sigma_e^2$$
$$x_6=M^2C_1^2(\theta)C_2'(\eta)P^2,\quad x_7=M^2C_1^2(\theta)C_2'(\eta)H^2 \tag{27}$$

$$\left[\frac{\partial^2 J_{VR}^{N}(\xi)}{\partial\theta^2}\right]_{\hat{\theta}_N}\to E\left[\frac{\partial^2 J_{VR}^{N}(\xi)}{\partial\theta^2}\right]_{\theta^*}=x_8Eu_L^2+x_9\sigma_e^2$$
$$x_8=\left(MC_1'(\theta)C_2(\eta)-C_1'(\theta)\right)P^2,\quad x_9=\left(MC_1'(\theta)C_2(\eta)-C_1'(\theta)\right)H^2 \tag{28}$$

Applying equation (26)、(27) and (28) into the matrix computation process, we derive that.

$$\begin{bmatrix}\dfrac{\partial^2 J_{VR}^{N}(\xi)}{\partial\theta^2} & \dfrac{\partial^2 J_{VR}^{N}(\xi)}{\partial\theta\partial\eta}\\ \dfrac{\partial^2 J_{VR}^{N}(\xi)}{\partial\eta\partial\theta} & \dfrac{\partial^2 J_{VR}^{N}(\xi)}{\partial\eta^2}\end{bmatrix}_{\hat{\xi}_N}\to\begin{bmatrix}x_8Eu_L^2+x_9\sigma_e^2 & x_4Eu_L^2+x_5\sigma_e^2\\ x_4Eu_L^2+x_5\sigma_e^2 & x_6Eu_L^2+x_7\sigma_e^2\end{bmatrix} \tag{29}$$

As the computation of asymptotic variance A needs to compute the inverse matrix with respect to equation (29), so we use the block matrix to get.

$$\begin{bmatrix}\dfrac{\partial^2 J_{VR}^{N}(\xi)}{\partial\theta^2} & \dfrac{\partial^2 J_{VR}^{N}(\xi)}{\partial\theta\partial\eta}\\ \dfrac{\partial^2 J_{VR}^{N}(\xi)}{\partial\eta\partial\theta} & \dfrac{\partial^2 J_{VR}^{N}(\xi)}{\partial\eta^2}\end{bmatrix}^{-1}=\frac{\begin{bmatrix}x_6Eu_L^2+x_7\sigma_e^2 & -\left(x_4Eu_L^2+x_5\sigma_e^2\right)\\ -\left(x_4Eu_L^2+x_5\sigma_e^2\right) & x_8Eu_L^2+x_9\sigma_e^2\end{bmatrix}}{\left(x_8Eu_L^2+x_9\sigma_e^2\right)\left(x_6Eu_L^2+x_7\sigma_e^2\right)-\left(x_4Eu_L^2+x_5\sigma_e^2\right)^2} \tag{30}$$

After substituting equation (25) and (30) into the computation of asymptotic variance matrix A, we establish the important asymptotic variance matrix expression with respect to these two controller parameter vectors $\xi=(\theta,\eta)$

$$A=\frac{\begin{bmatrix} x_6Eu_L^2+x_7\sigma_e^2 & -\left(x_4Eu_L^2+x_5\sigma_e^2\right) \\ -\left(x_4Eu_L^2+x_5\sigma_e^2\right) & x_8Eu_L^2+x_9\sigma_e^2 \end{bmatrix}}{\left[\left(x_8Eu_L^2+x_9\sigma_e^2\right)\left(x_6Eu_L^2+x_7\sigma_e^2\right)-\left(x_4Eu_L^2+x_5\sigma_e^2\right)^2\right]^2}\times\begin{bmatrix} x_1H^2Eu_L^2\sigma_e^2 & x_3 2PH^2Eu_L^2\sigma_e^2 \\ x_3 2PH^2Eu_L^2\sigma_e^2 & x_2H^2Eu_L^2\sigma_e^2 \end{bmatrix}$$
$$\times\begin{bmatrix} x_6Eu_L^2+x_7\sigma_e^2 & -\left(x_4Eu_L^2+x_5\sigma_e^2\right) \\ -\left(x_4Eu_L^2+x_5\sigma_e^2\right) & x_8Eu_L^2+x_9\sigma_e^2 \end{bmatrix}^T \tag{31}$$

Rewriting the product of three matrices in equation (31), we see that.

$$\begin{bmatrix} x_6 & -x_4 & x_7 & -x_5 \\ -x_4 & x_8 & -x_5 & x_9 \end{bmatrix}\begin{bmatrix} Eu_L^2 & 0 \\ 0 & Eu_L^2 \\ \sigma_e^2 & 0 \\ 0 & \sigma_e^2 \end{bmatrix}Eu_L^2H^2\sigma_e^2\times\begin{bmatrix} x_1 & x_3 2P \\ x_3 2P & x_2 \end{bmatrix}\begin{bmatrix} Eu_L^2 & 0 & \sigma_e^2 & 0 \\ 0 & Eu_L^2 & 0 & \sigma_e^2 \end{bmatrix}\begin{bmatrix} x_6 & -x_4 \\ -x_4 & x_8 \\ x_7 & -x_5 \\ -x_5 & x_9 \end{bmatrix}$$

Here we notice that in the asymptotic variance matrix A, the sub-matrix which position lies in block $(1,1)$ is the asymptotic variance matrix $\operatorname{cov}\left(\hat{\theta}_N\right)$ corresponding to unknown parameter vector θ. Meanwhile the sub-matrix which position lies in block $(2,2)$ is the asymptotic variance matrix $\operatorname{cov}(\hat{\eta}_N)$ corresponding to unknown parameter vector η. These two sub-matrices are all in the main diagonal. So here we only give the results about these two sub-matrices in the main diagonal and neglect other block sub-matrices. Through complex mathematical derivation, the final asymptotic variance matrix A is given as the following form.

$$\begin{aligned} A&=\begin{bmatrix} A_{11} & * \\ * & A_{22} \end{bmatrix}+\sigma_e^2\begin{bmatrix} \left(x_1x_7^2-4x_3x_5x_7P+x_2x_5^2\right) & * \\ * & \left(x_1x_5^2-4x_3x_5x_9P+x_2x_9^2\right) \end{bmatrix} \\ A_{11}&=\left(x_1x_6^2-4x_3x_4x_6P+x_2x_4^2\right)E^2u_L^2+x_{10}Eu_L^2 \\ A_{22}&=\left(x_1x_4^2-4x_3x_4x_8P+x_2x_8^2\right)E^2u_L^2+x_{11}Eu_L^2 \end{aligned} \tag{32}$$

In equation (32) x_{10} and x_{11} are two sums which consist four terms, they are expressed as.

$$\begin{aligned} x_{10}&=\left(2x_1x_6x_7-4x_3x_5x_6P+2x_2x_4x_5-4x_3x_4x_7P\right)\sigma_e^2 \\ x_{11}&=\left(2x_1x_4x_5-4x_3x_4x_9-2x_3x_5x_8+2x_2x_8x_9\right)\sigma_e^2 \end{aligned} \tag{33}$$

Based on equation (32) and (33), the asymptotic variance matrix with respect to those two controller parameter vectors in closed loop system is derived.

4. Optimal filter design

The optimal filter is always designed by minimizing one optimization problem. Here we choose the trace operation of that asymptotic variance matrix A as the objective cost in the optimization problem. Then we construct the following optimization problem.

$$\min_{L} \frac{\left(x_1x_6^2 - 4x_3x_4x_6P + x_2x_4^2 + x_1x_4^2 - 4x_3x_4x_8P + x_2x_8^2\right)E^2u_L^2}{\left[\left(x_8Eu_L^2 + x_9\sigma_e^2\right)\left(x_6Eu_L^2 + x_7\sigma_e^2\right) - \left(x_4Eu_L^2 + x_5\sigma_e^2\right)^2\right]^2} + \frac{\left(x_{10} + x_{11}\right)Eu_L^2 + \left(x_1x_7^2 - 4x_3x_5x_7P + x_2x_5^2x_1x_5^2 - 4x_3x_5x_9P + x_2x_9^2\right)\sigma_e^2}{\left[\left(x_8Eu_L^2 + x_9\sigma_e^2\right)\left(x_6Eu_L^2 + x_7\sigma_e^2\right) - \left(x_4Eu_L^2 + x_5\sigma_e^2\right)^2\right]^2} \tag{34}$$

Where in the numerator we suppose the pre-part of term $E^2u_L^2$ is b_2, the latter part of term $E^2u_L^2$ is b_1, and constant term is b_0. Similarly we take every factor in the expansion of denominator as a_2, a_1, a_0 respectively. So the optimization problem (34) can be simplified as that.

$$\min_{m} \frac{b_2m^2 + b_1m + b_0}{a_2m^2 + a_1m + a_0}, \quad m = Eu_L^2 \tag{35}$$

From the optimal necessary condition, we have.

$$\frac{\partial\left[\dfrac{b_2m^2 + b_1m + b_0}{a_2m^2 + a_1m + a_0}\right]}{\partial m} = 0 \tag{36}$$

Let that expansion of equation (36) equal to zero, then it will be satisfied that.

$$\left(a_1b_2 - a_2b_1\right)m^2 + \left(2a_0b_2 - 2a_2b_0\right)m + a_2b_1 - a_1b_0 = 0 \tag{37}$$

Using the formula of roots, we obtain the root of equation (37).

$$m = \frac{\left(a_0b_2 - a_2b_0\right) + \left[\left(a_0b_2 - a_2b_0\right)^2 - \left(a_1b_2 - a_2b_1\right)a_2b_1 - a_1b_0\right]^{\frac{1}{2}}}{\left(a_1b_2 - a_2b_1\right)} \tag{38}$$

As $u_L(t) = L(z)u(t)$, then we obtain.

$$Eu_L^2 = \left|L\left(e^{jw}\right)\right|^2 Eu^2 = m \tag{39}$$

The above equation (39) is one equality condition which the optimal filter would be satisfied.

5. Simulation example

Consider one discrete time linear system, this linear system is described as the following transfer function form.

$$P(z)=\frac{(z-1.2)(z-0.4)}{z(z-0.3)(z-0.8)}$$

The classical PID controller is used here in simulation.

$$C_1(\theta)=\alpha^T(z)\theta=\begin{bmatrix}\frac{z^2}{z^2-z} & \frac{z}{z^2-z} & \frac{1}{z^2-z}\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix}$$

The true PID controller is given as.

$$C_1(\theta)=\begin{bmatrix}\frac{z^2}{z^2-z} & \frac{z}{z^2-z} & \frac{1}{z^2-z}\end{bmatrix}\begin{bmatrix}0.86\\ 0.2\\ 0.1\end{bmatrix}$$

The expected closed loop transfer function is chosen as.

$$M(z)=\frac{z(z-1)\left(0.86z^4-1.1z^3+3.9z^2+0.8z+0.48\right)}{z^7-3z^6-0.96z^5-0.72z^4-0.93z^3+3.9z^2+0.8z+0.48}$$

The input-output measured data $\{u(t),y(t)\}_{t=1,2\cdots1000}$ are collected in the closed loop environment, the number of data points is set 2000. In order to use the idea of virtual reference feedback tuning control to design parameter vector $\hat{\theta}_N^T$, the plant model $P(z)$ is excited by zero mean Gaussian white noise with the number of points is 1000. The optimal filter $L(z)$ may be chosen as one form in equation (39). The nonlinear least squares algorithm is used to solve the optimization problem (8). Before starting this iteration algorithm, the initial values about the unknown parameter vector are selected as.

$$\theta=\begin{bmatrix}0.75 & 0.25 & 0.15\end{bmatrix}^T$$

In Fig 2, the asymptotic curves of three PID parameters in the unknown parameter vector $\hat{\theta}_N^T$ are plotted. The solid lines denote the sample variances which are computed by sample data, the dotted lines denote the asymptotic variances which are computed by using our asymptotic analysis formula. According to Fig 2, as the solid lines approximate to the dotted lines very closely, it means that the asymptotic value can be more close to the real sample value.

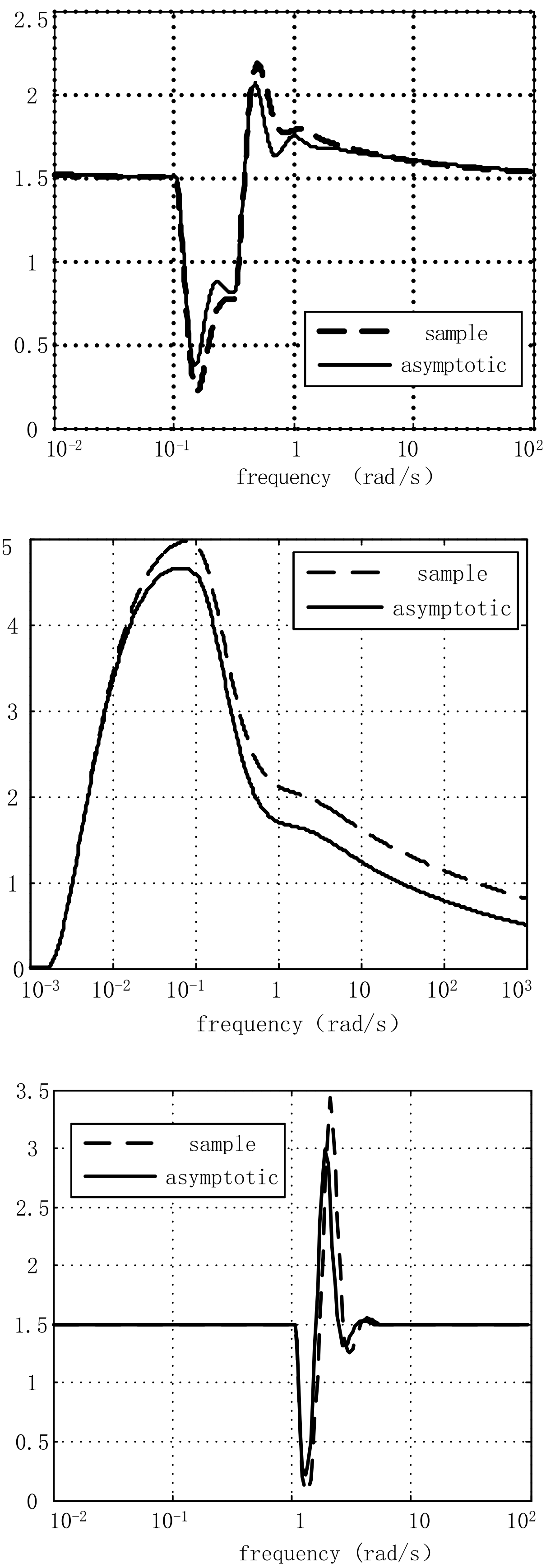

Fig 2: The asymptotic curves of three PID parameters

6. Conclusion

In this note, we derive the asymptotic variance matrix of the two unknown parameter vectors in the idea of virtual reference feedback tuning control. Then the optimal filter is designed on the basic of this asymptotic variance matrix. As the asymptotic variance matrix is more important in the latter input signal design and model validation process, so the next topic is about how to apply this asymptotic variance matrix to those two fields.

References

[1] Guido. Guardabassi. Virtual reference direct method: An off-line approach to data-based control system design[J]. IEEE Transactions of Automatic Control, 2002, 45(5): 954-960.

[2] M.C.Campi. Virtual reference feedback tuning: a direct method for the design of feedback controllers [J]. Automatica, 2002, 38(4): 1337-1346.

[3] M.C.Campi. A. Lechini. An application of the Virtual Reference Feedback Tuning method to a Benchmark problem[J]. European Journal of Control, 2003, 9(2):66-76.

[4] A.Lechini. M.C.Campi. Virtual reference feedback tuning for two degrees of freedom controllers [J]. International Journal of Adaptive Control and Signal Processing, 2002, 16(10): 355-371.

[5] M.C.Campi. Direct nonlinear control design: the virtual reference feedback tuning approach[J]. IEEE Transactions of Automatic Control, 2006, 51(1): 14-27.

[6] Alexandre S. Bazanella. Iterative minimization of H2 control performance criteria[J]. Automatica, 2008, 44(3): 2549-2559.

[7] R hilderbrand. Pre-filtering in iterative feedback tuning: optimization of the pre-filter for accuracy[J]. IEEE Transactions of Automatic Control, 2004, 49(8): 1801-1805.

[8] Vijay k Shetty. Priority based assignment and routing of unmanned combat aerial vehicles[J]. Computer & Operations Research, 2008, 35(10): 1813-1828.

[9] Silven S. A neural network based optimization algorithm for multiple targets tracking[J]. IEEE Journal of Oceanic Engineering, 2010, 37(4): 56-62.

[10] Zhang Wen xiu. Incomplete information system and its optimal selection [J]. Computer and Mathematics with Application, 2004, 48(5): 691-698.

[11] Wang Guo yin. Rough set extensions in incomplete information systems[J]. Frontiers of Electronical and Electronic Engineering, 2008, 3(4): 399-405.