SCIREA Journal of Computer
ISSN: 2995-6927
http://www.scirea.org/journal/Computer
June 17, 2025
Volume 10, Issue 2, April 2025
https://doi.org/10.54647/computer520447

How to Deduce 17 valid Non-trivial Generalized Modal Syllogisms from the Syllogism □EA◇H-2

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Abstract

A modern Square{Q}={Q, $Q\neg$, $\neg Q$, $\neg Q\neg$} is composed of the quantifier Q, its inner negation $Q\neg$, outer negation $\neg Q$ and dual negation $\neg Q\neg$. This paper focuses on studying the generalized modal syllogisms formed by modalities and the 8 quantifiers in Square{*no*} and Square{*at most half of the*}. More specifically, this paper firstly presents a knowledge representation of the syllogism □EA◇H-2, then proves its validity according to the truth value definitions of some categorical propositions. Finally, with the help of some reducible operations, the other 17 valid non-trivial generalized modal syllogisms can be obtained from the syllogism □EA◇H-2. This formal study is beneficial for knowledge mining in artificial intelligence.

Keywords: generalized modal syllogisms; validity; knowledge reasoning; knowledge mining

1. Introduction

The basis of generalized modal syllogistic[1-3] is generalized syllogistic[4] and classical modal syllogistic[5-6]. By adding at least one and at most three of non-overlapping necessary modalities ($\Box$) and/or possible ones ($\Diamond$) to a generalized syllogism, one can obtain a generalized modal syllogism[7]. Replacing classical quantifiers with generalized quantifiers in a classical modal syllogism, one can also obtain a generalized modal syllogism. Due to inconsistencies in study on classical modal syllogisms[8-9], there are only a few research results on generalized modal syllogisms.

In natural language, there are a great many generalized modal syllogisms, this paper only study the generalized modal syllogisms formed by two modalities ($\Box$ and $\Diamond$) and the 8 quantifiers in Square{*no*} and Square{*at most half of the*}. These quantifiers are the most common quantifiers in natural language.

2. Preliminaries

Assuming *g*, *z*, and *p* are lexical variables, and *D* is their domain in this paper. *G, Z*, and *P* are the sets composed of these three variables respectively. $|G\cap P|$ is the cardinality of the intersection *G* and *P.* Let α, η, μ, and θ be well-formed formulas (abbreviated as wff). '$\vdash \mu$' states that the wff μ is provable, '$\mu =_{def} \theta$' that μ can be defined by θ. The others are similar. 'if and only if' is abbreviated as 'iff'. The operators (such as, $\neg$, $\wedge$, $\rightarrow$, $\leftrightarrow$) are symbols in classical logic[10].

Let *Q* be a quantifier, and its inner, outer, and dual negative quantifier is respectively denoted as $Q\neg$, $\neg Q$, and $\neg Q\neg$. These four quantifiers form a modern Square{*Q*}={*Q*, $Q\neg$, $\neg Q$, $\neg Q\neg$}, in which any quantifier can define the other three quantifiers[1]. To be specific, in Square{*no*}, there are four classical quantifiers as follows: *no*, *all*, *not all, some*. In Square{*at most half of the*}, there are four non-trivial generalized quantifiers as follows: *at most half of the, at least half of the, most, fewer than half of the.* Statements formed by these 8 quantifiers are abbreviated as Proposition *E*, *A*, *O*, *I*, *H*, *S*, *M*, and *F* respectively. Therefore, the syllogism $\Box$EA$\Diamond$H-2 is the abbreviation of the second figure syllogism $\Box$*no(p, z)*$\wedge$*all(g, z)*$\rightarrow$$\Diamond$*at most half of the*(*g*, *p*).

Example 1:

Major premise: No dogs are necessarily pigs.

Minor premise: All the animals on this farm are pigs.

Conclusion: At most half of the the animals on this farm are possibly dogs.

If p, m, and g represent a dog, a pig, and an animal on this farm respectively, then the above syllogism can be formalized as '$\Box no(p, z) \wedge all(g, z) \rightarrow \Diamond at\ most\ half\ of\ the(g, p)$', which is denoted as □EA◇H-2.

3. Generalized Modal Syllogism System with the Quantifier from Square{*most*}

Since any quantifier in Square{*no*} can define the other three quantifiers, and this is similar to Square{*at most half of the*}, any quantifier in Square{*no*} and Square{*at most half of the*} can be chosen as the initial quantifiers. The initial quantifiers in this paper are *no* and *at most half of the.*

3.1 Primitive Symbols

(1) lexical variables: g, z, p;

(2) quantifiers: *no, at most half of the*;

(3) operators: $\neg$, $\rightarrow$, $\Box$;

(4) brackets: (,).

3.2 Formation Rules

(1) If Q is a quantifier, g and p are lexical variables, then $Q(g, p)$ is a wff;

(2) If α and μ are wffs, then so are $\neg\alpha$, $\alpha\rightarrow\mu$ and $\Box\alpha$;

(3) Only the sentences formed by the two rules are wffs.

3.3 Basic Axioms

A1: If α is a valid proposition, then $\vdash \alpha$;

A2: $\vdash \Box no(p, z) \wedge all(g, z) \rightarrow \Diamond at\ most\ half\ of\ the(g, p)$ (i.e. the syllogism □EA◇H-2).

3.4 Deductive Rules

Rule 1: From $\vdash (\alpha\rightarrow\eta)$ and $\vdash (\eta\wedge\mu\rightarrow\theta)$ infer $\vdash (\alpha\wedge\mu\rightarrow\theta)$;

Rule 2: From $\vdash (\eta\wedge\mu\rightarrow\theta)$ and $\vdash (\theta\rightarrow\alpha)$ infer $\vdash (\eta\wedge\mu\rightarrow\alpha)$;

Rule 3: From $\vdash (\eta\wedge\mu\rightarrow\theta)$ infer $\vdash (\neg\theta\wedge\eta\rightarrow\neg\mu)$

3.5 Relevant Definitions

D1: $(\alpha\wedge\mu)=_{\text{def}}\neg(\alpha\rightarrow\neg\mu)$;

D2: $(\alpha\leftrightarrow\mu)=_{\text{def}}(\alpha\rightarrow\mu)\wedge(\mu\rightarrow\alpha)$;

D3: $(Q\neg)(g, p)=_{\text{def}}Q(g, D-p)$;

D4: $(\neg Q)(g, p)=_{\text{def}}$ It is not the case that $Q(g, p)$;

D5: $\Diamond Q(g, p)=_{\text{def}}\neg\Box\neg Q(g, p)$;

D6: *all(g, p)* is true iff $G\subseteq P$ is true in any real world;

D7: *some(g, p)* is true iff $G\cap P\neq\varnothing$ is true in any real world;

D8: *no(g, p)* is true iff $G\cap P=\varnothing$ is true in any real world;

D9: *not all(g, p)* is true iff $G\nsubseteq P$ is true in any real world;

D10: *at most half of the(g, p)* is true iff $|G\cap P|\leq 0.5|G|$ is true in any real world;

D11: *at least half of the(g, p)* is true iff $|G\cap P|\geq 0.5|G|$ is true in any real world;

D12: *most(g, p)* is true iff $|G\cap P|>0.5|G|$ is true in any real world;

D13: *fewer than half of the(g, p)* is true iff $|G\cap P|<0.5|G|$ is true in any real world;

D14: $\Box$*no(g, p)* is true iff $G\cap P=\varnothing$ is true in any possible world;

D15: $\Diamond$*at most half of the(g, p)* is true iff $|G\cap P|\leq 0.5|G|$ is true in at least in one possible world.

3.6 Relevant Facts

Fact 1 (Inner Negation)

(1.1) $\vdash$ *all(g, p)*$\leftrightarrow$*no*$\neg$*(g, p)*;

(1.2) $\vdash$ *no(g, p)*$\leftrightarrow$*all*$\neg$*(g, p)*;

(1.3) $\vdash$ *some(g, p)*$\leftrightarrow$*not all*$\neg$*(g, p)*;

(1.4) $\vdash$ *not all(g, p)*$\leftrightarrow$*some*$\neg$*(g, p)*;

(1.5) $\vdash$ *most(g, p)*$\leftrightarrow$*fewer than half of the*$\neg$*(g, p)*;

(1.6) $\vdash$ *fewer than half of the(g, p)*$\leftrightarrow$*most*$\neg$*(g, p)*;

(1.7) $\vdash$ *at least half of the(g, p)*$\leftrightarrow$*at most half of the (g, p)*;

(1.8) ⊢ *at most half of the(g, p)*↔*at least half of the (g, p)*.

Fact 2 (Outer Negation)

(2.1) ⊢ ¬*all(g, p)*↔*not all(g, p)*;

(2.2) ⊢ ¬*not all(g, p)*↔*all(g, p)*;

(2.3) ⊢ ¬*no(g, p)*↔*some(g, p)*;

(2.4) ⊢ ¬*some(g, p)*↔*no(g, p)*;

(2.5) ⊢ ¬*most(g, p)*↔*at most half of the(g, p)*;

(2.6) ⊢ ¬*at most half of the(g, p)*↔*most(g, p)*;

(2.7) ⊢ ¬*fewer than half of the(g, p)*↔*at least half of the(g, p)*;

(2.8) ⊢ ¬*at least half of the(g, p)*↔*fewer than half of the(g, p)*.

Fact 3 (Symmetry):

(3.1) ⊢ *some(g, p)*↔*some(p, g)*;

(3.2) ⊢ *no(g, p)*↔*no(p, g)*.

Fact 4 (Subordination):

(4.1) ⊢ *all(g, p)*→*some(g, p)*;

(4.2) ⊢ *no(g, p)*→*not all(g, p)*;

(4.3) ⊢ *all(g, p)*→*most(g, p)*;

(4.4) ⊢ *most(g, p)*→*some(g, p)*;

(4.5) ⊢ *all(g, p)*→*at least half of the(g, p)*;

(4.6) ⊢ *at least half of the(g, p)*→*some(g, p)*;

(4.7) ⊢ *at least half of the(g, p)*→*most(g, p)*;

(4.8) ⊢ *at most half of the(g, p)*→*fewer than half of the(g, p)*;

(4.9) ⊢ *fewer than half of the(g, p)*→*not all(g, p)*;

(4.10) ⊢ *at most half of the(g, p)*→*not all(g, p)*;

(4.11) ⊢ *no(g, p)*→*fewer than half of the(g, p)*;

(4.12) ⊢ $no(g, p) \rightarrow at\ most\ half\ of\ the(g, p)$;

(4.13) ⊢ $\Box Q(g, p) \rightarrow Q(g, p)$;

(4.14) ⊢ $\Box Q(g, p) \rightarrow \Diamond Q(g, p)$;

(4.15) ⊢ $Q(g, p) \rightarrow \Diamond \mathrm{Q}(g, p)$.

Fact 5 (Dual):

(5.1) ⊢ $\neg\Diamond\neg Q(g, p) \leftrightarrow \Box Q(g, p)$;

(5.2) ⊢ $\neg\Box\neg Q(g, p) \leftrightarrow \Diamond Q(g, p)$;

(5.3) ⊢ $\neg\Box Q(g, p) \leftrightarrow \Diamond\neg Q(g, p)$;

(5.4) ⊢ $\neg\Diamond Q(g, p) \leftrightarrow \Box\neg Q(g, p)$.

The above facts are the fundamental knowledge in generalized modal syllogistic[11].

4. Knowledge Reasoning about Generalized Modal Syllogisms

If one syllogism can be inferred from another syllogism, one can say that the former has reducibility. According to the above definitions and facts, one can prove the following Theorem 1 and Theorem 2.

Theorem 1 (□EA◇H-2): The generalized modal syllogism $\Box no(p, z) \wedge all(g, z) \rightarrow \Diamond at\ most\ half\ of(g, p)$ is valid.

Proof: Suppose that $\Box no(p, z)$ and $all(g, z)$ are true, it follows that $P \cap Z = \varnothing$ is true in any possible world according to Definition D14, and $G \subseteq Z$ is true in any real world in terms of Definition D6. Because any real world is a possible world. It can be concluded that $|G \cap P| = \varnothing$ is true in any real world. Obviously, $|G \cap P| \leq 0.5|G|$ is true in any real world. That means that *at most half of(g, p)* is true in line with Definition D10. Therefore, $\Diamond at\ most\ half\ of(g, p)$ is true in the light of Fact (4.5), just as desired.

Theorem 2: The validity of the following 17 generalized modal syllogisms can be inferred from that of the syllogism □EA◇H-2:

(1) ⊢ □EA◇H-2→□EA◇H-1

(2) ⊢ □EA◇H-2→□EA◇F-2

(3) ⊢ □EA◇H-2→□EA◇F-2→□EA◇F-1

(4) ⊢ □EA◇H-2→A□M◇I-3

(5) ⊢ □EA◇H-2→A□M◇I-3→A□S◇I-3

(6) ⊢ □EA◇H-2→A□M◇I-3→□MA◇I-3

(7) ⊢ □EA◇H-2→A□M◇I-3→A□S◇I-3→□SA◇I-3

(8) ⊢ □EA◇H-2→□E□MO-1

(9) ⊢ □EA◇H-2→□E□MO-1→□E□SO-1

(10) ⊢ □EA◇H-2→□E□MO-1→□E□MO-2

(11) ⊢ □EA◇H-2→□E□MO-1→□E□SO-1→□E□SO-2

(12) ⊢ □EA◇H-2→□AE◇H-2

(13) ⊢ □EA◇H-2→□AE◇H-2→□AE◇H-4

(14) ⊢ □EA◇H-2→□AE◇H-2→□AE◇F-2

(15) ⊢ □EA◇H-2→□AE◇H-2→□AE◇H-4→□AE◇F-4

(16) ⊢ □EA◇H-2→□AE◇H-2→□AE◇F-2→□A□E◇F-2

(17) ⊢ □EA◇H-2→□AE◇H-2→□AE◇H-4→□AE◇F-4→□A□E◇F-4

Proof:

[1] ⊢ □*no(p, z)*∧*all(g, z)*→◇*at most half of the*(*g*, *p*) (i.e. □EA◇H-2, Basic Axiom A2)

[2] ⊢ □*no(z, p)*∧*all(g, z)*→◇*at most half of the*(*g*, *p*) (i.e. □EA◇H-1, by [1] and Fact (3.2))

[3] ⊢ □*no(p, z)*∧*all(g, z)*→◇*fewer than half of the*(*g*, *p*)

(i.e. □EA◇F-2, by [1], Fact (4.8) and Rule 2)

[4] ⊢ □*no(z, p)*∧ *all(g, z)*→◇*fewer than half of the*(*g*, *p*) (i.e. □EA◇F-1, by [3] and Fact (3.2))

[5] ⊢ ¬◇*at most half of the*(*g*, *p*)∧*all(g, z)*→¬□*no(p, z)* (by [1] and Rule 3)

[6] ⊢ □¬*at most half of the*(*g*, *p*)∧*all(g, z)*→◇¬*no(p, z)* (by [5], Fact (5.3) and (5.4))

[7] ⊢ □*most*(*g*, *p*)∧*all(g, z)*→◇*some(p, z)* (i.e. A□M◇I-3, by [6], Fact (2.3) and (2.6))

[8] ⊢ □*at least half of the*(*g*, *p*)∧*all(g, z)*→◇*some(p, z)*

(i.e. A□S◇I-3, by [7], Fact (4.7) and Rule 1)

[9] ⊢ □*most*(*g*, *p*)∧*all(g, z)*→◇*some(z, p)* (i.e. □MA◇I-3, by [7] and Fact (3.1))

[10] ⊢ □*at least half of the*(*g*, *p*)∧*all(g, z)*→◇*some(z, p)* (i.e. □SA◇I-3, by [8] and Fact (3.1))

[11] ⊢ ¬◇*at most half of the*(*g*, *p*)∧□*no(p, z)*→¬*all(g, z)* (by [1] and Rule 3)

[12] ⊢ $\Box\neg$*at most half of the*$(g, p)\wedge\Box$*no*$(p, z)\rightarrow\neg$*all*(g, z) (by [11] and Fact (5.4))

[13] ⊢ $\Box$*most*$(g, p)\wedge\Box$*no*$(p, z)\rightarrow$*not all*(g, z) (i.e. □E□MO-1, by [12], Fact (2.6) and (2.1))

[14] ⊢ $\Box$*at least half of the*$(g, p)\wedge\Box$*no*$(p, z)\rightarrow$*not all*(g, z)

(i.e. □E□SO-1, by [13], Fact (4.7) and Rule 1)

[15] ⊢ $\Box$*most*$(g, p)\wedge\Box$*no*$(z, p)\rightarrow$*not all*(g, z) (i.e. □E□MO-2, by [13] and Fact (3.2))

[16] ⊢ $\Box$*at least half of the*$(g, p)\wedge\Box$*no*$(z, p)\rightarrow$*not all*(g, z)

(i.e. □E□SO-2, by [14] and Fact (3.2))

[17] ⊢ $\Box$*all*$\neg(p, z)\wedge$*no*$\neg(g, z)\rightarrow\Diamond$*at most half of the*$(g, p)$ (by [1], Fact (1.2) and (1.1))

[18] ⊢ $\Box$*all*$(p, D-z)\wedge$*no*$(g, D-z)\rightarrow\Diamond$*at most half of the*$(g, p)$

(i.e. □AE◇H-2, by [17] and Definition D3)

[19] ⊢ $\Box$*all*$(p, D-z)\wedge$*no*$(D-z, g)\rightarrow\Diamond$*at most half of the*$(g, p)$

(i.e. □AE◇H-4, by [18] and Fact (3.2))

[20] ⊢ $\Box$*all*$(p, D-z)\wedge$*no*$(g, D-z)\rightarrow\Diamond$*fewer than half of the*$(g, p)$

(i.e. □AE◇F-2, by [18], Fact (4.8) and Rule 2)

[21] ⊢ $\Box$*all*$(p, D-z)\wedge$*no*$(D-z, g)\rightarrow\Diamond$*fewer than half of the*$(g, p)$

(i.e. □AE◇F-4, by [19], Fact (4.8) and Rule 2)

[22] ⊢ $\Box$*all*$(p, D-z)\wedge\Box$*no*$(g, D-z)\rightarrow\Diamond$*fewer than half of the*$(g, p)$

(i.e. □A□E◇F-2, by [20], Fact (4.13) and Rule 1)

[23] ⊢ $\Box$*all*$(p, D-z)\wedge\Box$*no*$(D-z, g)\rightarrow\Diamond$*fewer than half of the*$(g, p)$

(i.e. □A□E◇F-4, by [21], Fact (4.13) and Rule 1)

Theorem 2 illustrates that there are reducible relationships between/among the syllogism □EA◇H-2 and the 17 derived valid generalized modal syllogisms. Obviously, the proof processes in Theorem 2 have logical consistency.

5. Conclusion

This paper firstly presents a knowledge representation of the syllogism □EA◇H-2, then proves its validity according to the truth value definitions of some categorical propositions with the quantifier in Square{*no*} and Square{*at most half of the*}. Finally, with the help of some reducible operations, the other 17 valid non-trivial generalized modal syllogisms can be obtained from the syllogism □EA◇H-2. This formal study is beneficial for knowledge mining in artificial intelligence.

Acknowledgement

This work was supported by the National Social Science Fund of China under Grant No.24FZXB068.

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