SCIREA Journal of Computer

https://www.scirea.org/journal/Computer

March 9, 2020

Volume 5, Issue 1, February 2020

Reducible Relations between/among Aristotle's Modal Syllogisms①

Xiaojun Zhang

Institute of Logic and Information, Sichuan Normal University, Chengdu, China;

5 Jing'an Road, Jinjiang District, Chengdu, Sichuan, China

Email: 591551032@qq.com

Abstract

All valid Aristotle's modal syllogisms can be obtained by adding modal operators to 24 valid classical syllogisms. On the basis of the 20 valid modal syllogisms obtained by adding modal operators to valid classical syllogisms *AAA*-1 and *EAE*-1, this paper not only shows that the validity of the other 326 Aristotle's modal syllogisms can be derived by making full use of truth definition and symmetry of Aristotelian quantifiers in generalized quantifier theory, and propositional deformation rules in proof theory, but also shows that there are reducible relations between/among Aristotle's modal syllogisms. These innovative results are embodied in the 29 theorems proposed in this paper. The research methods used in the paper provide a simple and reasonable mathematical model to study generalized modal syllogisms. It is hoped that these innovative achievements will make contributions to further research on Aristotle's and generalized modal syllogistic logic, and to promote knowledge representation and

This work was supported by the National Social Science Foundation of China under Grant No. 19FZXB102.

knowledge reasoning in computer science, and natural language information processing.

Keywords: generalized quantifier theory; Aristotle's modal syllogisms; reducible relations; validity

1. Introduction

Syllogistic reasoning is the most intensively researched due to the role they have played in human reasoning from Aristotle onwards. The focus of Aristotle's discussion of syllogisms is Aristotle's system of modal syllogisms. It has been open to public inspection for over 2300 years, and has had many consistently bad reviews ([9], p.247). The prevailing view is that modal syllogsitic is incomprehensible due to its many faults and inconsistencies ([10], p.95).

The formal system, ŁA, proposed by Łukasiewicz ([5]) is designed to capture that Aristotle's judgments of "assertoric syllogism" are valid or invalid. Aristotle's modal syllogisms can not be expressed well in modal propositional logic with quantifiers, ŁM, developed by Łukasiewicz ([5]). McCall([`2], pp.31-32) points out that Łukasiewicz's modal system, ŁM, yields highly unAristotelian results. The L-X-M calculus, a formal system, developed by McCall(1963) has given any chance of matching Aristotle's judgments about which of the n-premised (for $n \geq 2$) "apodeictic syllogisms" are valid or not. Geach([3]) illustrates two approaches to understand Aristotle's work on modal syllogistic. The system proposed by Geach([4]) can not deal with the apodeictics syllogisms well. Johnson([6]) gives a semantics for McCall's L-X-M, and shows that asserted wffs in L-X-M are valid and rejected sentences are invalid. Johnson([7][8]) provides variants of McCall's L-X-M that are sound and complete system.

Benefiting from comments about McCall's L-X-M in Thom([15]) and Thomason([16][17]), Johnson([9]) turns to McCall's work on the "contingent syllogisms", and develops a modified system, QLXM'. The system provides formal countermodels for many invalid apodeictic, assertoric, or contingent syllogisms. Although interpreters have try to find some interpretation of the modal syllogistic so as to consistently cover the whole modal syllogistic developed, the outcomes of such attempts have been disappointing ([10], p.95). So Malink([10], [11]) try to find a consistent formal model for Aristotle's modal syllogistic.

Patterson([13]) tries to deal with Aristotle's modal logic through the eyes of modern modal monadic first order predicate logic. And classical syllogisms has already been researched from the perspective of generalized quantifier theory, such as Benthem([18]), Westerståhl([19]), and Zhang ([20][21][22]). It is natural to try to view Aristotle's modal logic through the eyes of modern modal logic, possible world semantics, and generalized quantifier theory. The paper attempts to do this. It is known that there are reducible relations between/among classical and generalized syllogisms ([21][22])). The other 22 valid classical syllogisms can be derived from the valid classical syllogisms 'Barbara' AAA-1 and 'Celarent' EAE-1 in the light of generalized quantifier theory([14]). The following paper sets out not only to show that the validity of one modal syllogism can be derived by the validity of another modal syllogism, but also to show that there are reducible relations between/among Aristotle's modal syllogisms. In this paper, reducible relation between two valid syllogisms means that one valid syllogism can be derived from another valid syllogism.

In the following paper, $\neg$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$, $\Box$, and $\Diamond$ are signs of negation, conjunction, conditionality, biconditionality, necessity, and possibility, respectively. Similar to classical syllogisms, a Aristotle's modal syllogism has two premises and one conclusion, and is a particular instantiation of a syllogistic scheme. An Aristotle's modal syllogism can be interpreted as the following example:

All animals are necessarily mortal.

<u>*All horses are animals.*</u>

Some horses necessarily are mortal.

The syllogism means that the one below the line can be derived from the sentences above the line semantically. It has the form $Q_1(M, P) \wedge Q_2(S, M) \Rightarrow Q_3(S, P)$, where S is the set of things or stuff that the subject term denotes, P is the set of things or stuff that the predicate term signifies, and M is the set of things or stuff that the middle term expresses, each of Q_1, Q_2, Q_3

in a Aristotle's modal syllogism is one of the following 12 generalized quantifiers *all*, *no*, *some*, *not all*, □*all*, □*no*, □*some*, □no*t all*, ◇*all*, ◇*no*, ◇*some* and ◇*not all*. In the above example, Q_1=□*all*, Q_2=*all*, and Q_3=□*some*, so the modal syllogism can be denoted as $\Box all(M, P) \wedge all(S, M) \Rightarrow \Box some(S, P)$. The other cases are similar.

2. Preliminaries

A quantified sentence including Aristotelian quantifiers '*Q S*s are *P*' is denoted by $Q(S, P)$, in which Q is one of Aristotelian quantifiers *all*, *some*, *no*, and *not all*. For example, a quantified sentence 'All children are eating' is denoted by $all(S, P)$, where S is the set of children in a given universe, P is the set of things that are eating in the universe, and the quantifier *all* is a relation between sets which is a particularly simple relation to describe: $S \subseteq P$.

Let S, P be an arbitrary set, the relations which Aristotelian quantifiers stand for can be given in standard set-theoretic notations as the following:

Definition 1:

(1) $all(S, P) \Leftrightarrow S \subseteq P$; (2) $no(S, P) \Leftrightarrow S \cap P = \varnothing$;

(3) $some(S, P) \Leftrightarrow S \cap P \neq \varnothing$; (4) $not\ all(S, P) \Leftrightarrow S - P \neq \varnothing$.

For the sake of simplicity, the universal affirmative and negative proposition '*All Ss are P*' and '*No Ss are P*' are denoted by $all(S, P)$ and $no(S, P)$, and abbreviated by A and I proposition, respectively. And the particular affirmative and negative proposition '*Some Ss are P*' and '*Not all Ss are P*' are denoted by $some(S, P)$ and $not\ all(S, P)$, and abbreviated by I and O proposition, respectively. The proposition '*All Ss are necessarily P*' and '*Some Ss are possibly P*' are denoted by $\Box all(S, P)$ and $\Diamond some(S, P)$, and abbreviated by $\Box A$ and $\Diamond I$ proposition, respectively. The other cases are similar.

Similar to classical syllogisms, modal syllogisms can be grouped into four different 'figures':

(1) first figure	(2) second figure	(3) third figure	(4) fourth figure
$Q_1(M, P)$	$Q_1(P, M)$	$Q_1(M, P)$	$Q_1(P, M)$

$\underline{Q_2(S, M)}$ $\underline{Q_2(S, M)}$ $\underline{Q_2(M, S)}$ $\underline{Q_2(M, S)}$

$Q_3(S, P)$ $Q_3(S, P)$ $Q_3(S, P)$ $Q_3(S, P)$

Here Q can be chosen among the following 12 generalized quantifiers *all*, *some*, *no*, *not all*, □*all*, □*some*, □*no*, □*not all*, ◇*all*, ◇*some*, ◇*no*, and◇*not all*, so there are 12×12×12×4−4×4×4×4 = 6656 Aristotle's modal syllogisms. Similar to classical syllogisms, a modal syllogism is valid if each instantiation of S, M and P verifying the premises also verifies the conclusion.

According to the above notation, in the third figure, if suppose that Q_1=□*all*, Q_2=□*some* and Q_3=◇*not all* and, then the modal syllogism □*all(M, P)* ∧ □*some(M, S)* ⇒◇*not all(S, P)* can be abbreviated by □*A*□*I*◇*O*-3. Similarly, the modal syllogism □*all(P, M)*∧*some(M, S)* ⇒ *some(S, P)* can be abbreviated by □*AII*-4. The other notations are similar to them.

In terms of modal logic ([1][2]), necessity is what is true at every possible world, and possibility is what is true at some. Specifically, let p be one of the proposition A, E, I, and O, it follows that:

Definition 2:

(1) □p is true just in case p itself is true at every possible world;

(2) ◇p is true just in case p itself is true at least one possible world.

According to Definition 1 and Definition 2, it follows that:

Definition 3:

(1) □*all(S, P)* is true just in case $S \subseteq P$ is true at every possible world.

(2) ◇*all(S, P)* is true just in case $S \subseteq P$ is true at least one possible world.

(3) □*some(S, P)* is true just in case $S \cap P \neq \varnothing$ is true at every possible world.

(4) ◇*some(S, P)* is true just in case $S \cap P \neq \varnothing$ is true at least one possible world.

(5) □*no(S, P)* is true just in case $S \cap P = \varnothing$ is true at every possible world.

(6) ◇*no(S, P)* is true just in case $S \cap P = \varnothing$ is true at least one possible world.

(7) □*not all(S, P)* is true just in case $S - P \neq \varnothing$ is true at every possible world.

(8) $\Diamond$*not all(S, P)* is true just in case $S{-}P \neq \varnothing$ is true at least one possible world.

According to Definition 3, it is clear that

Fact 1: Let p be one of the proposition A, E, I, and O

(1) $\Box p \Rightarrow p$; (2) $p \Rightarrow \Diamond p$; (3)$\Box p \Rightarrow \Diamond p$

For instance, $\Box A \Rightarrow A$, $A \Rightarrow \Diamond A$ and $\Box A \Rightarrow \Diamond A$. More specifically, the following Fact 2 holds.

Fact 2:

(1) $\Box$*all(S, P)* $\Rightarrow$ *all(S, P)*; (2) $\Box$*some(S, P)* $\Rightarrow$ *some(S, P)*;

(3) $\Box$*no(S, P)* $\Rightarrow$ *no(S, P)*; (4) $\Box$*not all(S, P)* $\Rightarrow$ *not all(S, P)*;

(5) *all(S, P)* $\Rightarrow$ $\Diamond$*all(S, P)*; (6) *some(S, P)* $\Rightarrow$ $\Diamond$*some(S, P)*;

(7) *no(S, P)* $\Rightarrow$ $\Diamond$*no(S, P)*; (8) *not all(S, P)* $\Rightarrow$ $\Diamond$*not all(S, P)*;

(9) $\Box$*all(S, P)* $\Rightarrow$ $\Diamond$*all(S, P)*; (10) $\Box$*some(S, P)* $\Rightarrow$ $\Diamond$*some(S, P)*;

(11) $\Box$*no(S, P)* $\Rightarrow$ $\Diamond$*no(S, P)*; (12) $\Box$*not all(S, P)* $\Rightarrow$ $\Diamond$*not all(S, P)*;

In the light of Definition 1 and Definition 3, it follows that

Fact 3:

(1) $\Box$*all(S, P)* $\Rightarrow$ $\Box$*some(S, P)*; (2) $\Diamond$*all(S, P)* $\Rightarrow$ $\Diamond$*some(S, P)*;

(2) $\Box$*no(S, P)* $\Rightarrow$ $\Box$*not all(S, P)*; (4) $\Diamond$*no(S, P)* $\Rightarrow$ $\Diamond$*not all(S, P)*;

(5) *all(S, P)* $\Rightarrow$ *some(S, P)* ; (6) *no(S, P)* $\Rightarrow$ *not all(S, P).*

According to Definition 1 and Definition 3, one can easily derive the following Fact 4:

Fact 4: Symmetry of *some* or *no*

(1) *some(S, P)* $\Leftrightarrow$ *some(P, S)*; (2) $\Box$*some(S, P)* $\Leftrightarrow$ $\Box$*some(P, S)*;

(2) $\Diamond$*some(S, P)* $\Leftrightarrow$ $\Diamond$*some(P, S)*; (4) n*o(S, P)* $\Leftrightarrow$ *no(P, S)*;

(5) $\Box$*no(S, P)* $\Leftrightarrow$ $\Box$*no(P, S)*; (6) $\Diamond$*no(S, P)* $\Leftrightarrow$ $\Diamond$*no(P, S)*;

For example, the proposition "*no dog is a fish*" is necessarily true, the proposition "*no fish is* *a dog*" is also necessarily true. According to Fact 4, replacing the statement part on the right proposition in a valid modal syllogism by the left, one will not change the validity of the obtained modal syllogism.

In the following paper, let α and β be a syllogism, α ⇒ β means that the syllogism β can be derived from the syllogism α, and one can say that there are reducible relations between the two syllogisms. For example, □*E*□*A*□*E*-1 ⇒ □*E*□*A*□*O*-1 means that □*E*□*A*□*O*-1 can be derived from □*E*□*A*□*E*-1. Hence, there are reducible relations between □*E*□*A*□*E*-1 and □*E*□*A*□*O*-1.

3. The Form Proof for Valid Modal Syllogisms

On the basis of set theory, generalized quantifier theory, and possible world semantics, which modal syllogisms are valid can be proved in the light of Definition 1, Definition 3 and Fact 2. Proofs for some of modal syllogisms can be easily constructed and will be omitted.

Theorem 1: The following 16 modal syllogisms are valid:

(1.1) [001]□*A*□*A*□*A*-1: □*all(M, P)*∧□*all(S, M)* ⇒ □*all(S, P)*

(1.2) [002]◇*A*◇*A*◇*A*-1: ◇*all(M, P)*∧◇*all(S, M)* ⇒ ◇*all(S, P)*

(1.3) [003]□*A*◇*A*◇*A*-1: □*all(M, P)*∧◇*all(S, M)* ⇒ ◇*all(S, P)*

(1.4) [004]◇*A*□*A*◇*A*-1: ◇*all(M, P)*∧□*all(S, M)* ⇒ ◇*all(S, P)*

(1.5) [005]□*AA*□*A*-1: □*all(M, P)*∧*all(S, M)* ⇒ □*all(S, P)*

(1.6) [006]*A*□*A*□*A*-1: *all(M, P)*∧□*all(S, M)* ⇒ □*all(S, P)*

(1.7) [007]◇*AA*◇*A*-1: ◇*all(M, P)*∧*all(S, M)* ⇒ ◇*all(S, P)*

(1.8) [008]*A*◇*A*◇*A*-1: *all(M, P)*∧◇*all(S, M)* ⇒ ◇*all(S, P)*

(1.9) [009]□*A*◇*AA*-1: □*all(M, P)*∧◇*all(S, M)* ⇒ *all(S, P)*

(1.10) [010]*AA*◇*A*-1: *all(M, P)*∧*all(S, M)* ⇒ ◇*all(S, P)*

Proof: For (1.1), suppose that □*all(M, P)* and □*all(S, M)* are true, then $M \subseteq P$ and $S \subseteq M$ is true at every possible world in terms of the clause (1) in Definition 3. Now it follows that $M \subseteq P$ and $S \subseteq M$, so it is clear that $S \subseteq P$ is true at every possible world. Therefore $\Box all(S, P)$ is true according to the clause (1) in Definition 3 again. This proves the claim that the Arisitotle's modal syllogism $\Box all(M, P) \wedge \Box all(S, M) \Rightarrow \Box all(S, P)$ is valid, just as required.

The other cases can be similarly proved as (1.1). For example, for (1.8), suppose that $all(M, P)$ and $\Diamond all(S, M)$ are true, then $all(M, P)$ is true if and only if $M \subseteq P$ is true in the light of the clause (1) in Definition 1, and it follows that $\Diamond all(S, M) \Leftrightarrow S \subseteq M$ at least one possible world by the clause (2) in Definition 3. Now it is easy to observe that $M \subseteq P$ and $S \subseteq M$ at least one possible world, hence $S \subseteq P$ is true at least one possible world. Thus $\Diamond all(S, P)$ is true in term of the clause (2) in Definition 3 again. Therefore $all(M, P) \wedge \Diamond all(S, M) \Rightarrow \Diamond all(S, P)$ is valid, as desired.

Theorem 2: The following 16 modal syllogisms are valid:

(2.1) [011]$\Box E \Box A \Box E$-1: $\Box no(M, P) \wedge \Box all(S, M) \Rightarrow \Box no(S, P)$

(2.2) [012]$\Diamond E \Diamond A \Diamond E$-1: $\Diamond no(M, P) \wedge \Diamond all(S, M) \Rightarrow \Diamond no(S, P)$

(2.4) [013]$\Box E \Diamond A \Diamond E$-1: $\Box no(M, P) \wedge \Diamond all(S, M) \Rightarrow \Diamond no(S, P)$

(2.5) [014]$\Diamond E \Box A \Diamond E$-1: $\Diamond no(M, P) \wedge \Box all(S, M) \Rightarrow \Diamond no(S, P)$

(2.7) [015]$\Box E A \Box E$-1: $\Box no(M, P) \wedge all(S, M) \Rightarrow \Box no(S, P)$

(2.8) [016]$E \Box A \Box E$-1: $no(M, P) \wedge \Box all(S, M) \Rightarrow \Box no(S, P)$

(2.9) [017]$\Diamond E A \Diamond E$-1: $\Diamond no(M, P) \wedge all(S, M) \Rightarrow \Diamond no(S, P)$

(2.10) [018]$E \Diamond A \Diamond E$-1: $no(M, P) \wedge \Diamond all(S, M) \Rightarrow \Diamond no(S, P)$

(2.11) [019]$\Box E \Diamond A E$-1: $\Box no(M, P) \wedge \Diamond all(S, M) \Rightarrow no(S, P)$

(2.16) [020]$E A \Diamond E$-1: $no(M, P) \wedge all(S, M) \Rightarrow \Diamond no(S, P)$

Proof: Theorem can be similarly proved as Theorem 1 by means of Definition 1, Definition 3 and Fact 2.

Theorem 3: **The validity of some of the following modal syllogisms can be derived by the validity of the other of the following modal syllogisms:**

(3.1) [001]$\Box A\Box A\Box A$-1 $\Rightarrow$ [021]$\Box A\Box A\Diamond A$-1

(3.2) [001]$\Box A\Box A\Box A$-1 $\Rightarrow$ [022]$\Box A\Box AA$-1

(3.3) [005]$\Box AA\Box A$-1 $\Rightarrow$ [023]$\Box AA\Diamond A$-1

(3.4) [005]$\Box AA\Box A$-1 $\Rightarrow$ [024]$\Box AAA$-1

(3.5) [006]$A\Box A\Box A$-1 $\Rightarrow$ [025]$A\Box A\Diamond A$-1

(3.6) [006]$A\Box A\Box A$-1 $\Rightarrow$ [026]$A\Box AA$-1

(3.7) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [027]$\Box E\Box A\Diamond E$-1

(3.8) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [028]$\Box E\Box AE$-1

(3.9) [015]$\Box EA\Box E$-1 $\Rightarrow$ [029]$\Box EA\Diamond E$-1

(3.10) [015]$\Box EA\Box E$-1 $\Rightarrow$ [030]$\Box EAE$-1

(3.11) [016]$E\Box A\Box E$-1 $\Rightarrow$ [031]$E\Box A\Diamond E$-1

(3.12) [016]$E\Box A\Box E$-1 $\Rightarrow$ [032]$E\Box AE$-1

Proof: From (3.1) to (3.6) in Theorem 3 can be easily deducible from Theorem 1 and Fact 2. From (3.7) to (3.12) in Theorem 1 can be derived from Theorem 2 and Fact 2.

Theorem 3 not only means that the 12 derived syllogisms are valid, but also means that reducible relations between these Aristotle's modal syllogisms. It is easily observed that the 20 valid Aristotle's modal syllogisms in Theorem 1 and Theorem 2 can be obtained by adding modal operators to valid classical syllogisms *AAA*-1 and *EAE*-1. In fact, all valid Aristotle's modal syllogisms can be obtained by adding modal operators to 24 valid classical syllogisms (for details, see [23]). It is clear that the number of all valid Aristotle's modal syllogisms obtained by adding modal operators to the valid classical syllogism *AAA*-1 is 16, that is, the ten syllogisms in Theorems 1 and the first six derived syllogisms in Theorem 3. And the other cases are similar, therefore the number of all valid Aristotle's modal syllogisms is 16×24=384.

4. **Reducible Relations between/among Aristotle's Modal Syllogisms:**

Xiaojun Zhang and Sheng Li (2016) derives the other 22 valid classical syllogisms from the valid classical syllogisms *AAA*-1 and *EAE*-1, that is, there are reducible relations between/among valid classical syllogisms. On the basis of the 20 valid modal syllogisms in Theorem 1 and Theorem 2, this paper not only shows that the validity of the other 326 Aristotle's modal syllogisms can be derived by making full use of truth definition and symmetry of Aristotelian quantifiers in generalized quantifier theory and propositional deformation rules in proof theory, but also shows that there are reducible relations between/among Aristotle's modal syllogisms.

In order to study reducible relations between modal syllogisms, the following two propositional deformation rules are required.

Propositional Deformation Rule 1: Let p, q, r be a proposition, $((\neg r\wedge p) \rightarrow \neg q)$ can be derived from $((p\wedge q) \rightarrow r)$.

Proof: From $((p\wedge q)\rightarrow r)$ it can be derived $(\neg r \rightarrow \neg(p\wedge q))$, then can be derived $(\neg r \rightarrow (\neg p \vee\neg q))$, and then can be derived $(\neg r \rightarrow (p\rightarrow\neg q))$, therefore it can can be derived $((\neg r\wedge p) \rightarrow \neg q)$. In short, from $((p\wedge q)\rightarrow$ r) it can be derived $((\neg r\wedge p) \rightarrow \neg q)$, as desired.

Propositional Deformation Rule 2: Let p, q, r be a proposition, $((\neg r\wedge p) \rightarrow \neg p)$ can be derived from $((p\wedge q) \rightarrow r)$.

Proof: From $((p\wedge q) \rightarrow r)$ it can be derived $((q\wedge p) \rightarrow r)$, then can be derived $(\neg r \rightarrow (\neg q\vee\neg p))$ and then can be derived $(\neg r \rightarrow (\neg q\vee\neg p))$, therefore it can can be derived $((\neg r\wedge q) \rightarrow \neg p)$. In short, from $((p\wedge q)\rightarrow r)$ it can be derived $((\neg r\wedge p) \rightarrow \neg p)$, just as required.

The proof of the following four theorems will use these two propositional deformation rules.

Theorem 4: [001] □*A*□*A*□*A*-1 ⇒ [067]◇*E*□*A*◇*O*-1

Proof: Since □*A* ⇒ □*I*, it is clear that [259] □*A*□*A*□*I*-4 can be derived from [001] □*A*□*A*□*A*-1. That is, □*all(P, M)*∧□*all(M, S)*⇒□*some(S, P)*. According to the Propositional Deformation Rule 2, it can be known that $((\neg r\wedge p) \rightarrow \neg p)$ can be derived from $((p\wedge q) \rightarrow r)$. Therefore, ¬□*some(S, P)*∧□*all(M, S)* ⇒ ¬□*all(P, M)* can be derived from □*all(P,*

$M) \wedge \Box all(M, S) \Rightarrow \Box some(S, P)$. Therefore, $\Diamond \neg some(S, P) \wedge \Box all(M, S) \Rightarrow \Diamond \neg all(P, M)$. And since $\neg some = no$ and $\neg all = not\ all$, $\Diamond no(S, P) \wedge \Box all(M, S) \Rightarrow \Diamond not\ all(P, M)$. By replacing S by M, P by S and M by P, it can be obtained that $\Diamond no(M, S) \wedge \Box all(P, M) \Rightarrow \Diamond not\ all(S, P)$. That is, the first figure modal syllogism [067]$\Diamond E \Box A \Diamond O$-1 is valid.

Theorem 5: [001] $\Box A \Box A \Box A$-1 ⇒ [169]$\Box A \Diamond O \Diamond O$-2

Proof: For [001] $\Box A \Box A \Box A$-1, i.e. $\Box all(M, P) \wedge \Box all(S, M) \Rightarrow \Box all(S, P)$. According to the Propositional Deformation Rule 1, it can be known that $((\neg r \wedge p) \rightarrow \neg q)$ can be derived from $((p \wedge q) \rightarrow r)$. Therefore, $\neg \Box all(S, P) \wedge \Box all(M, P) \Rightarrow \neg \Box all(S, M)$ can be derived from $\Box all(M, P) \wedge \Box all(S, M) \Rightarrow \Box all(S, P)$. Hence $\Diamond \neg all(S, P) \wedge \Box all(M, P) \Rightarrow \Diamond \neg all(S, M)$. And since $\neg all = not\ all$, $\Diamond not\ all(S, P) \wedge \Box all(M, P) \Rightarrow \Diamond not\ all(S, M)$. By replacing P by M, and M by P, it can be followed that $\Diamond not\ all(S, M) \wedge \Box all(P, M) \Rightarrow \Diamond not\ all(S, P)$. By putting the major premise in front of the minor premise it can be obtained that $\Box all(P, M) \wedge \Diamond not\ all(S, M) \Rightarrow \Diamond not\ all(S, P)$. In other words, the second figure modal syllogism [169]$\Box A \Diamond O \Diamond O$-2 is valid.

Theorem 6: [001] $\Box A \Box A \Box A$-1 ⇒ [247]$\Diamond O \Box A \Diamond O$-3

Proof: For [001] $\Box A \Box \mathrm{A} \Box \mathrm{A}$-1, i.e. $\Box all(M, P) \wedge \Box all(S, M) \Rightarrow \Box all(S, P)$. According to the Propositional Deformation Rule 2, it can be known that $((\neg r \wedge p) \rightarrow \neg p)$ can be derived from $((p \wedge q) \rightarrow r)$. Therefore, $\neg \Box all(S, P) \wedge \Box all(S, M) \Rightarrow \neg \Box all(M, P)$ can be inferred from $\Box all(M, P) \wedge \Box all(S, M) \Rightarrow \Box all(S, P)$. Since $\Box$ and $\Diamond$ can be mutually defined, $\Diamond \neg all(S, P) \wedge \Box all(S, M) \Rightarrow \Diamond \neg all(M, P)$. And since $\neg all = not\ all$, $\Diamond not\ all(S, P) \wedge \Box all(S, M) \Rightarrow \Diamond not\ all(M, P)$. By changing the letters it can be obtained that $\Diamond not\ all(M, P) \wedge \Box all(M, S) \Rightarrow \Diamond not\ all(S, P)$. That is, the third figure modal syllogism [247]$\Diamond O \Box A \Diamond O$-3 is valid.

Theorem 7: [001] $\Box A \Box A \Box A$-1⇒[307]$\Diamond E \Box A \Diamond O$-4

Proof: It is clear that [259]$\Box A \Box A \Box I$-4 can be derived from [001]$\Box A \Box A \Box A$-1. That is, $\Box all(P, M) \wedge \Box all(M, S) \Rightarrow \Box some(S, P)$. According to the Propositional Deformation Rule 1, it can be known that $((\neg r \wedge p) \rightarrow \neg q)$ can be derived from $((p \wedge q) \rightarrow r)$. Therefore, $\neg \Box some(S, P) \wedge \Box all(P, M) \Rightarrow \neg \Box all(M, S)$ can be derived from $\Box all(P, M) \wedge \Box all(M, S) \Rightarrow \Box some(S, P)$. And $\Box$ and $\Diamond$ can be mutually defined, hence $\Diamond \neg some(S, P) \wedge \Box all(P, M) \Rightarrow \Diamond \neg all(M, S)$.

And since $\neg$*some*= *no* and $\neg$*all*=*not all*, $\Diamond no(S, P) \wedge \Box all(P, M) \Rightarrow \Diamond not\ all(M, S)$. By replacing P by M, M by S and S by P it can be obtained that $\Diamond no(P, M) \wedge \Box all(M, S) \Rightarrow \Diamond not\ all(S, P)$. In other words, the fourth figure modal syllogism [307]$\Diamond E\Box A\Diamond O$-4 is valid.

It can be seen from the above four theorems that the valid modal syllogism of the first, second, third and fourth figures can be derived from the valid modal syllogism of the first figure. Now one can begin to study reducible relations between/among modal syllogisms on the basis of Definition 1, Definition 1, Fact 1, Fact 2, Propositional Deformation Rule 1 and Rule 2.

Theorem 8: Reducible relations between the following modal syllogisms:

(8.1) [001]$\Box A\Box A\Box A$-1 $\Rightarrow$ [033]$\Box A\Box A\Box I$-1

(8.2) [001]$\Box A\Box A\Box A$-1 $\Rightarrow$ [034]$\Box A\Box A\Diamond I$-1

(8.3) [001]$\Box A\Box A\Box A$-1 $\Rightarrow$ [035]$\Box A\Box AI$-1

(8.4) [002]$\Diamond A\Diamond A\Diamond A$-1 $\Rightarrow$ [036]$\Diamond A\Diamond A\Diamond I$-1

(8.5) [003]$\Box A\Diamond A\Diamond A$-1 $\Rightarrow$ [037]$\Box A\Diamond A\Diamond I$-1

(8.6) [004]$\Diamond A\Box A\Diamond A$-1 $\Rightarrow$ [038]$\Diamond A\Box A\Diamond I$-1

(8.7) [005]$\Box AA\Box A$-1 $\Rightarrow$ [039]$\Box AA\Box I$-1

(8.8) [005]$\Box AA\Box A$-1 $\Rightarrow$ [040]$\Box AA\Diamond I$-1

(8.9) [005]$\Box AA\Box A$-1 $\Rightarrow$ [041]$\Box AAI$-1

(8.10) [006]$A\Box A\Box A$-1 $\Rightarrow$ [042]$A\Box A\Box I$-1

(8.11) [006] $A\Box A\Box A$-1 $\Rightarrow$ [043]$A\Box A\Diamond I$-1

(8.12) [006] $A\Box A\Box A$-1 $\Rightarrow$ [044]$A\Box AI$-1

(8.13) [007]$\Diamond AA\Diamond A$-1 $\Rightarrow$ [045]$\Diamond AA\Diamond I$-1

(8.14) [008] $A\Diamond A\Diamond A$-1 $\Rightarrow$ [046]$A\Diamond A\Diamond I$-1

(8.15) [009]$\Box A\Diamond AA$-1 $\Rightarrow$ [047]$\Box A\Diamond AI$-1

(8.16) [010]$AA\Diamond A$-1 $\Rightarrow$ [048]$AA\Diamond I$-1

Proof: Theorem 8 can be deduced from Theorem 1 by use of Fact 2. It is known that a necessarily particular affirmative proposition can be implied by a necessarily universal affirmative proposition. Specifically, for (8.1), according to the clause (13) in Fact 2, it is clear that $\Box A \Rightarrow \Box I$, therefore [033]$\Box A\Box A\Box I$-1 can be obtained by replacing the conclusion $\Box A$ in [001]$\Box A\Box A\Box A$-1 with $\Box I$. The others can be similarly proved by means of Fact 2.

Theorem 9: Reducible relations between the following modal syllogisms:

(9.1) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [049]$\Box A\Box I\Diamond I$-1

(9.2) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [050]$\Box A\Diamond I\Diamond I$-1

(9.3) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [051]$\Box AI\Diamond I$-1

(9.4) [013]$\Box E\Diamond A\Diamond E$-1 $\Rightarrow$ [052]$\Box A\Box I\Box I$-1

(9.5) [014]$\Diamond E\Box A\Diamond E$-1 $\Rightarrow$ [053]$\Diamond A\Box I\Diamond I$-1

(9.6) [015]$\Box EA\Box E$-1 $\Rightarrow$ [054]$\Box A\Box$II-1

(9.7) [015]$\Box EA\Box E$-1 $\Rightarrow$ [055]$\Box AII$-1

(9.8) [015]$\Box EA\Box E$-1 $\Rightarrow$ [056]$\Box A\Diamond II$-1

(9.9) [016]$E\Box A\Box E$-1 $\Rightarrow$ [057]$A\Box II$-1

(9.10) [016]$E\Box A\Box E$-1 $\Rightarrow$ [058]$A\Diamond I\Diamond I$-1

(9.11) [016]$E\Box A\Box E$-1 $\Rightarrow$ [059]$A\Box I\Diamond I$-1

(9.12) [019]$\Box E\Diamond AE$-1 $\Rightarrow$ [060]$\Box AI\Box I$-1

(9.13) [020]$EA\Diamond E$-1 $\Rightarrow$ [061]$A\Box II$-1

Proof: Theorem 9 can be similarly proved in terms of the symmetry of *no*, Propositional Deformation Rule 1, and Fact 2. More specifically, for (9.1). According to the symmetry of *no,* [091]$\Box A\Box E\Box E$-2 can be obtained by replacing $\Box$no(S, P) in the conclusion of [011]$\Box E\Box A\Box E$-1 with $\Box$no(P, S). And [092] $\Box A\Box E\Diamond E$-2 can certainly be deduced from [091]$\Box A\Box E\Box E$-2 in view of that $\Box E \Rightarrow \Diamond E$. Then [049]$\Box A\Box I\Diamond I$-1 can be deduced from [092]$\Box A\Box E\Diamond E$-2 in the light of Propositional Deformation Rule 1, just as proved in Theorem 5 and Theorem 7.

For (9.4). According to the symmetry of *no,*[095]$\Box A\Diamond E\Diamond E$-2 can be obtained by replacing the conclusion □*no(S, P)* in [013]□*E*◇*A*◇*E*-1 with □*no(P, S)*. By taking advantage of Propositional Deformation Rule 1, one can transform [095]□*A*◇*E*◇*E*-2 to obtain [052]□*A*□*I*□*I*-1, just as proved in Theorem 5 and Theorem 7. The other arguments are analogous to that given for (9.1) and (9.4) just proved or easier.

It can be seen from the above proof process in (9.1) that [011]□*E*□*A*□*E*-1 ⇒ [091]□*A*□*E*□*E*-2 ⇒ [092] □*A*□*E*◇*E*-2 ⇒ [049]□*A*□*I*◇*I*-1. That not only means that the three modal syllogisms derived from [011]□*E*□*A*□*E*-1 are valid, but also means that there are reducible relations among the four syllogisms. The other cases are similar.

Theorem 10: Reducible relations between the following modal syllogisms:

(10.1) [011]□*E*☑*A*□*E*-1 ⇒ [062]□*E*□*A*□*O*-1

(10.2) [011]□*E*□*A*□*E*-1 ⇒ [063]□*E*□*A*◇*O*-1

(10.3) [011]□*E*□*A*□*E*-1 ⇒ [064]□*E*□*AO*-1

(10.4) [012]◇*E*◇*A*◇*E*-1 ⇒ [065]◇*E*◇*A*◇*O*-1

(10.5) [013]□*E*◇*A*◇*E*-1 ⇒ [066]□*E*◇*A*◇*O*-1

(10.6) [014]◇*E*□*A*◇*E*-1 ⇒ [067]◇*E*□*A*◇*O*-1

(10.7) [015]□*EA*□*E*-1 ⇒ [068]□*EA*◇*O*-1

(10.8) [015]□*EA*□*E*-1 ⇒ [069]□*EAO*-1

(10.9) [015]□*EA*□*E*-1 ⇒ [070]□*EA*□*O*-1

(10.10) [016]*E*□*A*□*E*-1 ⇒ [071]*E*□*A*□*O*-1

(10.11) [016]*E*□*A*□*E*-1 ⇒ [072]*E*□*A*◇*O*-1

(10.12) [016]*E*□*A*□*E*-1 ⇒ [073]*E*□*AO*-1

(10.13) [017]◇*EA*◇*E*-1 ⇒ [074]◇*EA*◇*O*-1

(10.14) [018]*E*◇*A*◇*E*-1 ⇒ [075]*E*◇*A*◇*O*-1

(10.15) [019]□*E*◇*AE*-1 ⇒ [076]□*E*◇*AO*-1

(10.16) [020]*EA*◇*E*-1 ⇒ [077]*EA*◇*O*-1

Proof: Similar to Theorem 8, Theorem 10 can be easily deduced from Fact 2. It is clear that a necessarily particular affirmative proposition can be certainly implied by a necessarily universal affirmative proposition. Specifically, for (10.1), □E⇒□O according to the clause (15) in Fact 2, therefore [062] □E□A□O-1 can certainly be obtained by replacing the conclusion □A in [011]□E□A□E-1 with .□O The other argument is analogous to that given for (10.1) by means of Fact 1.

Theorem 11: Reducible relations between the following modal syllogisms:

(11.1) [011]□*E*□*A*□*E*-1 ⇒ [078]□*E*□*I*◇*O*-1

(11.2) [011]□*E*□*A*□*E*-1 ⇒ [079]□*E*◇*I*◇*O*-1

(11.3) [011]□*E*□*A*□*E*-1 ⇒ [080]□*EI*◇*O*-1

(11.4) [013]□*E*◇*A*◇*E*-1 ⇒ [081]□*E*□*I*□*O*-1

(11.5) [014]◇*E*□*A*◇*E*-1 ⇒ [082]◇*E*□*I*◇*O*-1

(11.6) [015]□*EA*□*E*-1 ⇒ [083]□*E*□*IO*-1

(11.7) [015]□*EA*□*E*-1 ⇒ [084]□*E*◇*IO*-1

(11.8) [015]□*EA*□*E*-1 ⇒ [085]□*EIO*-1

(11.9) [016]*E*□*A*□*E*-1 ⇒ [086]*E*◇*I*◇*O*-1

(11.10) [016]*E*□*A*□*E*-1 ⇒ [087]*E*□*I*◇*O*-1

(11.11) [016]*E*□*A*□*E*-1 ⇒ [088]*EI*◇*O*-1

(11.12) [019]□*E*◇*AE*-1 ⇒ [089]□*EI*□*O*-1

(11.13) [020]*EA*◇*E*-1 ⇒ [090]*E*□*IO*-1

Proof: Similar to Theorem 9, Theorem 11 can be followed from the symmetry of *no*, Propositional Deformation Rule 1, and Fact 2. More specifically, for (11.1), in the light of the symmetry of *no,* [123]□*E*□*A*□*E*-2 can be obtained by replacing the major premise □*no(M, P)* in[011]□*E*□*A*□*E*-1 with □*no(P, M)*. According to [123]□*E*□*A*□*E*-2 and that □*E*⇒◇*E*, each one can easily derive [124] □*E*□*A*◇*E*-2. And then [078]□*E*□*I*◇*O*-1 can be obtained from [124]□*E*□*A*◇*E*-2 according to Propositional Deformation Rule 1, just as proved in

Theorem 5 and Theorem 7 . The other proofs are analogous to that given for (11.1) just proved or easier.

Theorem 12: Reducible relations between the following modal syllogisms:

(12.1) [011]□E□A□E-1 ⇒ [091]□A□E□E-2

(12.2) [011]□E□A□E-1 ⇒ [092]□A□E◇E-2

(12.3) [011]□E□A□E-1 ⇒ [093]□A□EE-2

(12.4) [012]◇E◇A◇E-1 ⇒ [094]◇A◇E◇E-2

(12.5) [013]□E◇A◇E-1 ⇒ [095]□A◇E◇E-2

(12.6) [014]◇E□A◇E-1 ⇒ [096]◇A□E◇E-2

(12.7) [015]□EA□E-1 ⇒ [097]□AE□E-2

(12.8) [015]□EA□E-1 ⇒ [098]□AE◇E-2

(12.9) [015]□EA□E-1 ⇒ [099]□AEE-2

(12.10) [016]E□A□E-1 ⇒ [100]A□E□E-2

(12.11) [016]E□A□E-1 ⇒ [101]A□E◇E-2

(12.12) [016]E□A□E-1 ⇒ [102]A□EE-2

(12.13) [017]◇EA◇E-1 ⇒ [103]◇AE◇E-2

(12.14) [018]E◇A◇E-1 ⇒ [104]A◇E◇E-2

(12.15) [019]□E◇AE-1 ⇒ [105]□A◇EE-2

(12.16) [020]EA◇E-1 ⇒ [106]AE◇E-2

Proof: Theorem 12 can be similarly proved by means of the symmetry of *no* and Fact 2. More specifically, for (12.2), according to the symmetry of *no,* [091]□A□E□E-2 can be obtained by substituting □ *no(P, S)* for the conclusion □ *no(S, P)* in [011] □ E □ A □ E-1. And [092]□A□E◇E-2 can be shown from [091]□A□E□E-2, since □E ⇒ ◇E in terms of the clause (12) in Fact 2. The other verifications follow the same pattern as the worked example above.

Theorem13: Reducible relations between the following modal syllogisms:

(13.1) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [107]$\Box A\Box E\Box O$-2

(13.2) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [108]$\Box A\Box E\Diamond O$-2

(13.3) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [109]$\Box A\Box EO$-2

(13.4) [012]$\Diamond E\Diamond A\Diamond E$-1 $\Rightarrow$ [110]$\Diamond A\Diamond E\Diamond O$-2

(13.5) [013]$\Box E\Diamond A\Diamond E$-1 $\Rightarrow$ [111]$\Box A\Diamond E\Diamond O$-2

(13.6) [014]$\Diamond E\Box A\Diamond E$-1 $\Rightarrow$ [112]$\Diamond A\Box E\Diamond O$-2

(13.7) [015]$\Box EA\Box E$-1 $\Rightarrow$ [113]$\Box AE\Box O$-2

(13.8) [015]$\Box EA\Box E$-1 $\Rightarrow$ [114]$\Box AE\Diamond O$-2

(13.9) [015]$\Box EA\Box E$-1 $\Rightarrow$ [115]$\Box AEO$-2

(13.10) [016]$E\Box A\Box E$-1 $\Rightarrow$ [116]$A\Box E\Box O$-2

(13.11) [016]$E\Box A\Box E$-1 $\Rightarrow$ [117]$A\Box E\Diamond O$-2

(13.12) [016]$E\Box A\Box E$-1 $\Rightarrow$ [118]$A\Box EO$-2

(13.13) [017]$\Diamond EA\Diamond E$-1 $\Rightarrow$ [119]$\Diamond AE\Diamond O$-2

(13.14) [018]$E\Diamond A\Diamond E$-1 $\Rightarrow$ [120]$A\Diamond E\Diamond O$-2

(13.15) [019]$\Box E\Diamond AE$-1 $\Rightarrow$ [121]$\Box A\Diamond EO$-2

(13.16) [020]$EA\Diamond E$-1 $\Rightarrow$ [122]$AE\Diamond O$-2

Proof: Similar to Theorem 12, Theorem 13 can be followed from the symmetry of *no* and Fact 2. More specifically, for (13.5), in the light of the symmetry of *no,* [095]$\Box A\Diamond E\Diamond E$-2 can be deduced by replacing the conclusion $\Box$*no(S, P)* in [013]$\Box$E$\Diamond$A$\Diamond$E-1 with $\Box$*no(P, S)*. It is clear that $\Diamond E \Rightarrow \Diamond O$ from Fact 2, and hence [111]$\Box A\Diamond E\Diamond O$-2 is deducible from [095]$\Box A\Diamond E\Diamond E$-2, just as required. The others are similar or easier to prove.

Theorem 14: Reducible relations between the following modal syllogisms:

(14.1) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [123]$\Box E\Box A\Box E$-2

(14.2) [011]□E□A□E-1 ⇒ [124]□E□A◇E-2

(14.3) [011]□E□A□E-1 ⇒ [125]□E□AE-2

(14.4) [012]◇E◇A◇E-1 ⇒ [126]◇E◇A◇E-2

(14.5) [013]□E◇A◇E-1 ⇒ [127]□E◇A◇E-2

(14.6) [014]◇E□A◇E-1 ⇒ [128]◇E□A◇E-2

(14.7) [015]□EA□E-1 ⇒ [129]□EA□E-2

(14.8) [015]□EA□E-1 ⇒ [130]□EA◇E-2

(14.9) [015]□EA□E-1 ⇒ [131]□EAE-2

(14.10) [016]E□A□E-1 ⇒ [132]E□A□E-2

(14.11) [016]E□A□E-1 ⇒ [133]E□A◇E-2

(14.12) [016]E□A□E-1 ⇒ [134]E□AE-2

(14.13) [017]◇EA◇E-1 ⇒ [135]◇EA◇E-2

(14.14) [018]E◇A◇E-1 ⇒ [136]E◇A◇E-2

(14.15) [019]□E◇AE-1 ⇒ [137]□E◇AE-2

(14.16) [020]EA◇E-1 ⇒ [138]EA◇E-2

Proof: Similar to Theorem 12 and Theorem 13, Theorem 14 can be followed from the symmetry of *no* and Fact 2. Specifically, for (14.8), in terms of the symmetry of *no,* [129]□EA□E-2 can be included by substituting □*no(P, M)* for the major premise □*no(M, P)* in [015] □ EA □ E-1. Since □ E ⇒ ◇ E, [130] □ EA ◇ E-2 can be deducible from [129]□EA□E-2, as desired. The other verifications are analogous to that given for (14.8) just proved or easier .

Theorem 15: Reducible relations between the following modal syllogisms:

(15.1) [011]□E□A□E-1 ⇒ [139]□E□A□O-2

(15.2) [011]□E□A□E-1 ⇒ [140]□E□A◇O-2

(15.3) [011]□E□A□E-1 ⇒ [141]□E□AO-2

(15.4) [012]◇*E*◇*A*◇*E*-1 ⇒ [142]◇*E*◇*A*◇*O*-2

(15.5) [013]□*E*◇*A*◇*E*-1 ⇒ [143]□*E*◇*A*◇*O*-2

(15.6) [014]◇*E*□*A*◇*E*-1 ⇒ [144]◇*E*□*A*◇*O*-2

(15.7) [015]□*EA*□*E*-1 ⇒ [145]□*EA*□*O*-2

(15.8) [015]□*EA*□*E*-1 ⇒ [146]□*EA*◇*O*-2

(15.9) [015]□*EA*□*E*-1 ⇒ [147]□*EAO*-2

(15.10) [016]*E*□*A*□*E*-1 ⇒ [148]*E*□*A*□*O*-2

(15.11) [016]E□A□E-1 ⇒ [149]*E*□*A*◇*O*-2

(15.12) [016]*E*□*A*□*E*-1 ⇒ [150]*E*□*AO*-2

(15.13) [017]◇*EA*◇*E*-1 ⇒ [151]◇*EA*◇*O*-2

(15.14) [018]*E*◇*A*◇*E*-1 ⇒ [152]*E*◇*A*◇*O*-2

(15.15) [019]□*E*◇*AE*-1 ⇒ [153]□*E*◇*AO*-2

(15.16) [020]*EA*◇*E*-1 ⇒ [154]*EA*◇*O*-2

Proof: Similar to Theorem 12-14, Theorem 15 can be followed from the symmetry of *no* and Fact 2. More specifically, for (15.12), [072]*E*□*A*◇*O*-1 can be obtained from [016]*E*□*A*□*E*-1 and □*E* ⇒*O*. And then [150]*E*□*AO*-2 can be derived by replacing the major premise *no(M, P)* in [072]*E*□*A*◇*O*-1 with *no(P, M)*, just as required. The other proofs are similar to that given for (15.12) just proved or easier .

Theorem 16: Reducible relations between the following modal syllogisms:

(16.1) [011]□*E*□*A*□*E*-1 ⇒ [155]□*E*□*I*◇*O*-2

(16.2) [011]□*E*□*A*□*E*-1 ⇒ [156]□*E*◇*I*◇*O*-2

(16.3) [011]□*E*□*A*□*E*-1 ⇒ [157]□*EI*◇*O*-2

(16.4) [013]□*E*◇*A*◇*E*-1 ⇒ [158]□*E*□*I*□*O*-2

(16.5) [014]◇*E*□*A*◇*E*-1 ⇒ [159]◇*E*□*I*◇*O*-2

(16.6) [015]$\Box EA\Box E$-1 $\Rightarrow$ [160]$\Box E\Box IO$-2

(16.7) [015]$\Box EA\Box E$-1 $\Rightarrow$ [161]$\Box EIO$-2

(16.8) [016]$E\Box A\Box E$-1 $\Rightarrow$ [162]$E\Diamond I\Diamond O$-2

(16.9) [016]$E\Box A\Box E$-1 $\Rightarrow$ [163]$\Box E\Diamond IO$-2

(16.10) [016]$E\Box A\Box E$-1 $\Rightarrow$ [164]$\Box E\Box IO$-2

(16.11) [016]$E\Box A\Box E$-1 $\Rightarrow$ [165]$EI\Diamond O$-2

(16.12) [019] $\Box E\Diamond AE$-1 $\Rightarrow$ [166]$\Box EI\Box O$-2

(16.13) [020] $AA\Diamond A$-1 $\Rightarrow$ [167]$E\Box IO$-2

Proof: Similar to Theorem 9 and Theorem 11, Theorem 16 can be obtained according to Propositional Deformation Rule 1 and Fact 2. More specifically, for (16.12), [029]$\Box EA\Diamond E$-1 can be deduced from [015]$\Box EA\Box E$-1 and $\Box E \Rightarrow\Diamond E$. Then [166]$\Box E\Box IO$-2 can be obtained by transforming [029]$\Box EA\Diamond E$-1 by means of Propositional Deformation Rule 1, just as proved in Theorem 5 and Theorem 7. The other verifications are similar to that given for (16.12) just proved or easier.

Theorem 17: Reducible relations between the following modal syllogisms:

(17.1) [001]$\Box A\Box A\Box A$-1 $\Rightarrow$ [168]$\Box A\Box O\Diamond O$-2

(17.2) [001]$\Box A\Box A\Box A$-1 $\Rightarrow$ [169]$\Box A\Diamond O\Diamond O$-2

(17.3) [001]$\Box A\Box A\Box A$-1 $\Rightarrow$ [170]$\Box AO\Diamond O$-2

(17.4) [003]$\Box A\Diamond A\Diamond A$-1 $\Rightarrow$ [171]$\Box A\Box O\Box O$-2

(17.5) [004]$\Diamond A\Box A\Diamond A$-1 $\Rightarrow$ [172]$\Diamond A\Box O\Diamond O$-2

(17.6) [005]$\Box AA\Box A$-1 $\Rightarrow$ [173]$\Box A\Box OO$-2

(17.7) [005]$\Box AA\Box A$-1 $\Rightarrow$ [174]$\Box AOO$-2

(17.8) [006]$A\Box A\Box A$-1 $\Rightarrow$ [175]$A\Diamond O\Diamond O$-2

(17.9) [006]$A\Box A\Box A$-1 $\Rightarrow$ [176]$\Box A\Diamond OO$-2

(17.10) [006]$A\Box A\Box A$-1 $\Rightarrow$ [177]$\Box A\Box OO$-2

(17.11) [006]$A\Box A\Box A$-1 $\Rightarrow$ [178]A$O\Diamond O$-2

(17.12) [009] $\Box A\Diamond AA$-1 $\Rightarrow$ [179]$\Box AO\Box O$-2

(17.13) [010] $AA\Diamond A$-1 $\Rightarrow$ [180]$A\Box OO$-2

Proof: Similar to Theorem 9, 11 and 16, Theorem 17 can be obtained from Propositional Deformation Rule 1 and Fact 2. More specifically, for (17.1), [021]$\Box A\Box A\Diamond A$-1 can be deduced from [001]$\Box A\Box A\Box A$-1 and $\Box A \Rightarrow \Diamond A$. And [168]$\Box A\Box O\Diamond O$-2 can be obtained from [021]$\Box A\Box A\Diamond A$-1 in views of Propositional Deformation Rule 1, just as proved in Theorem 5 and Theorem 7. The other arguments are analogous to that given for (17.1), just proved.

Theorem 18: Reducible relations between the following modal syllogisms:

(18.1) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [181]$\Box A\Box I\Diamond I$-3

(18.2) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [182]$\Box A\Diamond I\Diamond I$-3

(18.3) [011]$\Box EA\Box E$-1 $\Rightarrow$ [183]$\Box AI\Diamond I$-3

(18.4) [013]$\Box E\Diamond A\Diamond E$-1 $\Rightarrow$ [184]$\Diamond A\Box I\Diamond I$-3

(18.5) [014]$\Diamond E\Box A\Diamond E$-1 $\Rightarrow$ [185]$\Box A\Box I\Box I$-3

(18.6) [015]$\Box EA\Box E$-1 $\Rightarrow$ [186]$A\Diamond I\Diamond I$-3

(18.7) [015]$\Box EA\Box E$-1 $\Rightarrow$ [187]$AI\Diamond I$-3

(18.8) [015]$\Box EA\Box E$-1 $\Rightarrow$ [188]$A\Box I\Diamond I$-3

(18.9) [016]$E\Box A\Box E$-1 $\Rightarrow$ [189]$\Box A\Diamond II$-3

(18.10) [016]$E\Box A\Box E$-1 $\Rightarrow$ [190]$\Box A\Box II$-3

(18.11) [016]$E\Box A\Box E$-1 $\Rightarrow$ [191]$\Box AII$-3

(18.12) [019]$\Box E\Diamond AE$-1 $\Rightarrow$ [192]$\Diamond AI\Diamond I$-3

(18.13) [020]$EA\Diamond E$-1 $\Rightarrow$ [193]$A\Box II$-3

Proof: Theorem 18 can be deducible from the symmetry of *no*, Fact 2 and Propositional Deformation Rule 2. Specifically, for (18.3), with reference to the symmetry of *no*,

[123]□*E*□*A*□*E*-2 can be obtained by substituting □*no(P, M)* for the major premise □*no(M, P)* in [011]□*EA*□*E*-1. And [125]□*E*□*AE*-2 can be derived from [123]□*E*□*A*□*E*-2 and □*E* ⇒*E*. Then [183]□*AI*◇*I*-3 can be followed by transforming [125]□*E*□*AE*-2 according to Propositional Deformation Rule 2, just as proved in Theorem 4 and Theorem 6. The other verifications are similar to that given for (18.3) just proved or easier.

Theorem 19: Reducible relations between the following modal syllogisms:

(19.1) [011]□*E*□*A*□*E*-1 ⇒ [194]□*A*□*A*◇*I*-3

(19.2) [011]□*E*□*A*□*E*-1 ⇒ [195]□*A*◇*A*◇*I*-3

(19.3) [011]□*E*□*A*□*E*-1 ⇒ [196]□*AA*◇*I*-3

(19.4) [013]□*E*◇*A*◇*E*-1 ⇒ [197]◇*A*□*A*◇*I*-3

(19.5) [014]◇*E*□*A*◇*E*-1 ⇒ [198]□*A*□*A*□*I*-3

(19.6) [015]□*EA*□*E*-1 ⇒ [199]*A*◇*A*◇*I*-3

(19.7) [015]□*EA*□*E*-1 ⇒ [200]*A*□*A*◇*I*-3

(19.8) [015]□*EA*□*E*-1 ⇒ [201]*AA*◇*I*-3

(19.9) [016]*E*□*A*□*E*-1 ⇒ [202]□*A*□*AI*-3

(19.10) [016]*E*□*A*□*E*-1 ⇒ [203]□*A*◇*AI*-3

(19.11) [016]*E*□*A*□*E*-1 ⇒ [204]□*AAI*-3

(19.12) [019]□*E*◇*AE*-1 ⇒ [205]◇*AA*◇*I*-3

(19.13) [020]*EA*◇*E*-1 ⇒ [206]*A*□*AI*-3

Proof: Similar to Theorem 18, Theorem 19 can be followed from the symmetry of *no*, Fact 2 and Propositional Deformation Rule 2. More specifically, for (19.12), [076]□*E*◇*AO*-1 can be deduced from [019]□*E*◇*AE*-1 and *E* ⇒ *O*. [153]□*E*◇*AO*-2 can be obtained by replacing the major premise □*no(M, P)* in [011]□*EA*□*E*-1 with □*no(P, M)*. And then [205]◇*AA*◇*I*-3 can be deduced from [125]□*E*□*AE*-2 in views of Propositional Deformation Rule 2, just as proved in Theorem 4 and Theorem 6. The other verifications follow the same pattern as the just worked example.

Theorem 20: Reducible relations between the following modal syllogisms:

(20.1) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [207]$\Box E\Box A\Diamond O$-3

(20.2) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [208]$\Box E\Diamond A\Diamond O$-3

(20.3) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [209]$\Box EA\Diamond O$-3

(20.4) [013]$\Box E\Diamond A\Diamond E$-1 $\Rightarrow$ [210]$\Diamond E\Box A\Diamond O$-3

(20.5) [014]$\Diamond E\Box A\Diamond E$-1 $\Rightarrow$ [211]$\Box E\Box A\Box O$-3

(20.6) [015]$\Box EA\Box E$-1 $\Rightarrow$ [212]$E\Diamond A\Diamond O$-3

(20.7) [015]$\Box EA\Box E$-1 $\Rightarrow$ [213]$E\Box A\Diamond O$-3

(20.8) [015]$\Box EA\Box E$-1 $\Rightarrow$ [214]$EA\Diamond O$-3

(20.9) [016]$E\Box A\Box E$-1 $\Rightarrow$ [215]$\Box E\Box AO$-3

(20.10) [016]$E\Box A\Box E$-1 $\Rightarrow$ [216]$\Box E\Diamond AO$-3

(20.11) [016]$E\Box A\Box E$-1 $\Rightarrow$ [217]$\Box EAO$-3

(20.12) [019]$\Box E\Diamond AE$-1 $\Rightarrow$ [218]$\Diamond EA\Diamond O$-3

(20.13) [020]$EA\Diamond E$-1 $\Rightarrow$ [219]$E\Box AO$-3

Proof: Similar to Theorem 18 and Theorem 19, Theorem 20 can be derived from the symmetry of *no*, Fact 2 and Propositional Deformation Rule 2. Specifically, for (20.8), [097]$\Box AE\Box E$-2 can be obtained by substituting $\Box no(P, S)$ for the conclusion $\Box no(S, P)$ in [015]$\Box EA\Box E$-1. And [115]$\Box AEO$-2 can be followed from [097]$\Box AE\Box E$-2 and $\Box E \Rightarrow O$. And then [214] $EA\Diamond O$-3 can be deduced by from [115] $\Box AEO$-2 with reference to Propositional Deformation Rule 2, just as proved in Theorem 4 and Theorem 6. The other cases are similar.

Theorem 21: Reducible relations between the following modal syllogisms:

(21.1) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [220]$\Box E\Box I\Diamond O$-3

(21.2) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [221]$\Box E\Diamond I\Diamond O$-3

(21.3) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [222]$\Box EI\Diamond O$-3

(21.4) [013]□*E*◇*A*◇*E*-1 ⇒ [223]◇*E*□*I*◇*O*-3

(21.5) [014]◇*E*□*A*◇*E*-1 ⇒ [224]□*E*□*I*□*O*-3

(21.6) [015]□*EA*□*E*-1 ⇒ [225]*E*◇*I*◇*O*-3

(21.7) [015]□*EA*□*E*-1 ⇒ [226]*E*□*I*◇*O*-3

(21.8) [015]□*EA*□*E*-1 ⇒ [227]*EI*◇*O*-3

(21.9) [016]*E*□*A*□*E*-1 ⇒ [228]□*E*□*IO*-3

(21.10) [016]*E*□*A*□*E*-1 ⇒ [229]□*E*◇*IO*-3

(21.11) [016]*E*□*A*□*E*-1 ⇒ [230]□*EIO*-3

(21.12) [019]□*E*◇*AE*-1 ⇒ [231]◇*EI*◇*O*-3

(21.13) [020]*EA*◇*E*-1 ⇒ [232]*E*□*IO*-3

Proof: Similar to Theorem 18-20, Theorem 21 can be followed from the symmetry of *no*, Fact 2 and Propositional Deformation Rule 2. More specifically, for (21.7), [097]□*AE*□*E*-2 can be obtained by replacing the conclusion □*no(S, P)* in [015]□*EA*□*E*-1 with □*no(P, S)*. And [098] □*AE*◇*E*-2 can be deducible from [097]□*AE*□*E*-2 and □*E*⇒◇*E*. And then [226] *E*□*I*◇*O*-3 can be derived from [098]□*AE*◇*E*-2 in terms of Propositional Deformation Rule 2, just as proved in Theorem 4 and Theorem 6. The other arguments are similar or easier.

Theorem 22: Reducible relations between the following modal syllogisms:

(22.1) [011]□*E*□*A*□*E*-1 ⇒ [233]□*I*□*A*◇*I*-3

(22.2) [011]□*E*□*A*□*E*-1 ⇒ [234]◇*I*□*A*◇*I*-3

(22.3) [011]□*E*□*A*□*E*-1 ⇒ [235]□*IA*◇*I*-3

(22.4) [013]□*E*◇*A*◇*E*-1 ⇒ [236]□*I*◇*A*◇*I*-3

(22.5) [014]◇*E*□*A*◇*E*-1 ⇒ [237]□*I*□*A*□*I*-3

(22.6) [015]□*EA*□*E*-1 ⇒ [238]*I*◇*A*◇*I*-3

(22.7) [015]□*EA*□*E*-1 ⇒ [239]*I*□*A*◇*I*-3

(22.8) [015]□*EA*□*E*-1 ⇒ [240]*IA*◇*I*-3

(22.9) [016]$E\square A\square E$-1 $\Rightarrow$ [241]$\square I\square AI$-3

(22.10) [016]$E\square A\square E$-1 $\Rightarrow$ [242]$\square I\Diamond AI$-3

(22.11) [016]$E\square A\square E$-1 $\Rightarrow$ [243]$\square IAI$-3

(22.12) [019]$\square E\Diamond AE$-1 $\Rightarrow$ [244]$\square IA\Diamond I$-3

(22.13) [020]$EA\Diamond E$-1 $\Rightarrow$ [245]$I\square AI$-3

Proof: Similar to Theorem 9, 11 and 16-17, Theorem 22 can be obtained from Fact 2 and Propositional Deformation Rule 2. Specifically, for (22.9), [031]$E\square A\Diamond E$-1 can be derived from [016]$E\square A\square E$-1 and $\square E\Rightarrow\Diamond E$. And [241]$\square I\square AI$-3 can be deducible from [031]$E\square A\Diamond E$-1 by means of Propositional Deformation Rule 2, just as proved in Theorem 4 and Theorem 6. The other arguments are analogous to that given for (22.5) just proved or easier.

Theorem 23: Reducible relations between the following modal syllogisms:

(23.1) [001]$\square A\square A\square A$-1 $\Rightarrow$ [246]$\square O\square A\Diamond O$-3

(23.2) [001]$\square A\square A\square A$-1 $\Rightarrow$ [247]$\Diamond O\square A\Diamond O$-3

(23.3) [001]$\square A\square A\square A$-1 $\Rightarrow$ [248]$\square OA\Diamond O$-3

(23.4) [003]$\square A\Diamond A\Diamond A$-1 $\Rightarrow$ [249]$\square O\Diamond A\Diamond O$-3

(23.5) [004]$\Diamond A\square A\Diamond A$-1 $\Rightarrow$ [250]$\square O\Diamond A\Diamond O$-3

(23.6) [005]$\square AA\square A$-1 $\Rightarrow$ [251]$O\Diamond A\Diamond O$-3

(23.7) [005]$\square AA\square A$-1 $\Rightarrow$ [252]$O\square A\Diamond O$-3

(23.8) [005]$\square AA\square A$-1 $\Rightarrow$ [253]$OA\Diamond O$-3

(23.9) [006]$A\square A\square A$-1 $\Rightarrow$ [254]$\square O\square AO$-3

(23.10) [006]$A\square A\square A$-1 $\Rightarrow$ [255]$\square O\Diamond AO$-3

(23.11) [006]$A\square A\square A$-1 $\Rightarrow$ [256]$\square OAO$-3

(23.12) [009]$\square A\Diamond AA$-1 $\Rightarrow$ [257]$\square OA\Diamond O$-3

(23.13) [010]$AA\Diamond A$-1 $\Rightarrow$ [258]$O\square AO$-3

Proof: Similar to Theorem 22, Theorem 23 can be obtained from Fact 2 and Propositional Deformation Rule 2. More specifically, for (23.7), [023]$\Box AA\Diamond A$-1 can be followed from [005]$\Box AA\Box A$-1 and $\Box A\Rightarrow\Diamond A$. And then [252]$O\Box A\Diamond O$-3 can be deducible from [023]$\Box AA\Diamond A$-1 by use of Propositional Deformation Rule 2, just as proved in Theorem 4 and Theorem 6. The other verifications are similar to that given for (23.7) just proved or easier.

Theorem 24: Reducible relations between the following modal syllogisms:

(24.1) [001] $\Box A\Box A\Box A$-1 $\Rightarrow$ [259]$\Box A\Box A\Box I$-4

(24.2) [001]$\Box A\Box A\Box A$-1 $\Rightarrow$ [260]$\Box A\Box A\Diamond I$-4

(24.3) [001]$\Box A\Box A\Box A$-1 $\Rightarrow$ [261]$\Box A\Box AI$-4

(24.4) [002]$\Diamond A\Diamond A\Diamond A$-1 $\Rightarrow$ [262]$\Diamond A\Diamond A\Diamond I$-4

(24.5) [003]$\Box A\Diamond A\Diamond A$-1 $\Rightarrow$ [263]$\Box A\Diamond A\Diamond I$-4

(24.6) [004]$\Diamond A\Box A\Diamond A$-1 $\Rightarrow$ [264]$\Diamond A\Box A\Diamond I$-4

(24.7) [005]$\Box AA\Box A$-1 $\Rightarrow$ [265]$\Box AA\Box I$-4

(24.8) [005]$\Box AA\Box A$-1 $\Rightarrow$ [366]$\Box AA\Diamond I$-4

(24.9) [005]$\Box AA\Box A$-1 $\Rightarrow$ [367]$\Box AAI$-4

(24.10) [006]$A\Box A\Box A$-1 $\Rightarrow$ [268]$A\Box A\Box I$-4

(24.11) [006]$A\Box A\Box A$-1 $\Rightarrow$ [269]$A\Box A\Diamond I$-4

(24.12) [006]$A\Box A\Box A$-1 $\Rightarrow$ [270]$A\Box AI$-4

(24.13) [007]$\Diamond AA\Diamond A$-1 $\Rightarrow$ [271]$\Diamond AA\Diamond I$-4

(24.14) [008]$A\Diamond A\Diamond A$-1 $\Rightarrow$ [272]$A\Diamond A\Diamond I$-4

(24.15) [009]$\Box A\Diamond AA$-1 $\Rightarrow$ [273]$\Box A\Diamond AI$-4

(24.16) [010]$AA\Diamond A$-1 $\Rightarrow$ [274]$AA\Diamond I$-4

Proof: Similar to Theorem 4, Theorem 24 can be deduced from the symmetry of *some*, Fact 2 and some transformations. More specifically, for (24.4), [037]$\Diamond A\Diamond A\Diamond I$-1 can be followed from [002]$\Diamond A\Diamond A\Diamond A$-1 and $\Box A \Rightarrow \Diamond I$. In fact, [037]$\Diamond A\Diamond A\Diamond I$-1 can be written as $\Diamond all(M, P)\wedge\Diamond all(S, M) \Rightarrow \Diamond some(S, P)$. And with reference to the symmetry of *some,* $\Diamond all(M, P)\wedge\Diamond all(S, M) \Rightarrow \Diamond some(P, S)$ can be obtained by replacing the conclusion $\Diamond some(S, P)$ in [037] $\Diamond A\Diamond A\Diamond I$-1 with $\Diamond some(P, S)$. By replacing *P* by *S*, and *S* by *P*, it can be obtained that $\Diamond all(M, S)\wedge\Diamond all(P, M) \Rightarrow \Diamond some(S, P)$ from $\Diamond all(M, P)\wedge\Diamond all(S, M) \Rightarrow \Diamond some(P, S)$. By putting the major premise in front of the minor premise it can be obtained that $\Diamond all(P, M)\wedge\Diamond all(M, S) \Rightarrow \Diamond some(S, P)$. In short, the fourth figure modal syllogism [262] $\Diamond A\Diamond A\Diamond I$-4 is valid, as required. The other verifications are similar to that given for (24.2) just proved or easier .

Theorem 25: Reducible relations between the following modal syllogisms:

(25.1) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [275]$\Box A\Box E\Box E$-4

(25.2) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [276]$\Box A\Box E\Diamond E$-4

(25.3) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [277]$\Box A\Box EE$-4

(25.4) [012]$\Diamond E\Diamond A\Diamond E$-1 $\Rightarrow$ [278]$\Diamond A\Diamond E\Diamond E$-4

(25.5) [013]$\Box E\Diamond A\Diamond E$-1 $\Rightarrow$ [279]$\Box A\Diamond E\Diamond E$-4

(25.6) [014]$\Diamond E\Box A\Diamond E$-1 $\Rightarrow$ [280]$\Diamond A\Box E\Diamond E$-4

(25.7) [015]$\Box EA\Box E$-1 $\Rightarrow$ [281]$\Box AE\Box E$-4

(25.8) [015]$\Box EA\Box E$-1 $\Rightarrow$ [282]$\Box AE\Diamond E$-4

(25.9) [015]$\Box EA\Box E$-1 $\Rightarrow$ [283]$\Box AEE$-4

(25.10) [016]$E\Box A\Box E$-1 $\Rightarrow$ [284]$A\Box E\Box E$-4

(25.11) [016]$E\Box A\Box E$-1 $\Rightarrow$ [285]$A\Box E\Diamond E$-4

(25.12) [016]$E\Box A\Box E$-1 $\Rightarrow$ [286]$A\Box EE$-4

(25.13) [017]$\Diamond EA\Diamond E$-1 $\Rightarrow$ [287]$\Diamond AE\Diamond E$-4

(25.14) [018]$E\Diamond A\Diamond E$-1 $\Rightarrow$ [288]$A\Diamond E\Diamond E$-4

(25.15) [019]$\Box E\Diamond AE$-1 $\Rightarrow$ [289]$\Box A\Diamond EE$-4

(25.16) [020]$EA\Diamond E$-1 $\Rightarrow$ [290]$AE\Diamond E$-4

Proof: Similar to Theorem 12-15, Theorem 25 can be followed from the symmetry of *no* and Fact 2. Specifically, for (25.3), [091]$\Box A\Box E\Box E$-2 can be obtained by substituting $\Box no(P, S)$ for the conclusion $\Box no(S, P)$ in [011]$\Box E\Box A\Box E$-1. It is clear that [093]$\Box A\Box EE$-2 can be deduced from [091]$\Box A\Box E\Box E$-2 and $\Box E \Rightarrow E$. And then [277]$\Box A\Box EE$-4 can be derived by replacing the minor premise $\Box no(S, M)$ in [093]$\Box A\Box EE$-2 with $\Box no(M, S)$, just as desired. The other arguments follow the same pattern as that given for (25.3) just proved or easier.

Theorem 26: Reducible relations between the following modal syllogisms:

(26.1) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [291]$\Box A\Box E\Box O$-4

(26.2) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [292]$\Box A\Box E\Diamond O$-4

(26.3) [011]$\Box E\Box A\Box E$-1 $\Rightarrow$ [293]$\Box A\Box EO$-4

(26.4) [012]$\Diamond E\Diamond A\Diamond E$-1 $\Rightarrow$ [294]$\Diamond A\Diamond E\Diamond O$-4

(26.5) [013]$\Box E\Diamond A\Diamond E$-1 $\Rightarrow$ [295]$\Box A\Diamond E\Diamond O$-4

(26.6) [014]$\Diamond E\Box A\Diamond E$-1 $\Rightarrow$ [296]$\Diamond A\Box E\Diamond O$-4

(26.7) [015]$\Box EA\Box E$-1 $\Rightarrow$ [297]$\Box AE\Box O$-4

(26.8) [015]$\Box EA\Box E$-1 $\Rightarrow$ [298]$\Box AE\Diamond O$-4

(26.9) [015]$\Box EA\Box E$-1 $\Rightarrow$ [299]$\Box AEO$-4

(26.10) [016]$E\Box A\Box E$-1 $\Rightarrow$ [300]$A\Box E\Box O$-4

(26.11) [016]$E\Box A\Box E$-1 $\Rightarrow$ [301]$A\Box E\Diamond O$-4

(26.12) [016]$E\Box A\Box E$-1 $\Rightarrow$ [302]$A\Box EO$-4

(26.13) [017]$\Diamond EA\Diamond E$-1 $\Rightarrow$ [303]$\Diamond AE\Diamond O$-4

(26.14) [018]$E\Diamond A\Diamond E$-1 $\Rightarrow$ [304]$A\Diamond E\Diamond O$-4

(26.15) [019]$\Box E\Diamond AE$-1 $\Rightarrow$ [305]$\Box A\Diamond EO$-4

(26.16) [020]$EA\Diamond E$-1 $\Rightarrow$ [306]$AE\Diamond O$-4

Proof: Similar to Theorem 12-15 and Theorem 25, Theorem 26 can be deduced from the symmetry of *no* and Fact 2. More specifically, for (26.12), [100]$A\Box E\Box E$-2 can be followed by substituting $\Box$*no(P, S)* for the conclusion $\Box$*no(S, P)* in [[016]$E\Box A\Box E$-1. And [118] $A\Box EO$-2 can be derived from [100]$A\Box E\Box E$-2 and $\Box E \Rightarrow O$. And then [302]$A\Box EO$-4 can be obtained by replacing the minor premise *no(S, M)* in [118]$A\Box EO$-2 with *no(M, S)*, just as required. The other proofs follow the same pattern as the worked example above.

Theorem 27: Reducible relations between the following modal syllogisms:

(27.1) [001]$\Box A\Box A\Box A$-1 $\Rightarrow$ [307] $\Box E\Box A\Diamond O$-4

(27.2) [001]$\Box A\Box A\Box A$-1 $\Rightarrow$ [308]$\Diamond E\Box A\Diamond O$-4

(27.3) [001]$\Box A\Box A\Box A$-1 $\Rightarrow$ [309]$E\Box A\Diamond O$-4

(27.4) [003]$\Box A\Diamond A\Diamond A$-1 $\Rightarrow$ [310]$\Box E\Box A\Box O$-4

(27.5) [004]$\Diamond A\Box A\Diamond A$-1 $\Rightarrow$ [311]$\Box E\Diamond A\Diamond O$-4

(27.6) [005]$\Box AA\Box A$-1 $\Rightarrow$ [312]$\Box E\Box AO$-4

(27.7) [005]$\Box AA\Box A$-1 $\Rightarrow$ [313]$E\Box AO$-4

(27.8) [006]$A\Box A\Box A$-1 $\Rightarrow$ [314]$\Diamond EA\Diamond O$-4

(27.9) [006]$A\Box A\Box A$-1 $\Rightarrow$ [315]$\Box EA\Diamond O$-4

(27.10) [006]$A\Box A\Box A$-1 $\Rightarrow$ [316]$EA\Diamond O$-4

(27.11) [007]$\Diamond AA\Diamond A$-1 $\Rightarrow$ [317]$\Box E\Diamond AO$-4

(27.12) [008]$A\Diamond A\Diamond A$-1 $\Rightarrow$ [318]$\Box EA\Box O$-4

(27.13) [009]$\Box A\Diamond AA$-1 $\Rightarrow$ [319]$E\Box A\Box O$-4

(27.14) [010]$AA\Diamond A$-1 $\Rightarrow$ [320]$\Box EAO$-4

Proof: Similar to Theorem 18-21, Theorem 27 can be followed from the symmetry of *some*, Fact 2 and Propositional Deformation Rule 1. Specifically, for (27.12), it is clear that [046]$A\Diamond A\Diamond I$-1 can be deducible from [010] $A\Diamond A\Diamond A$-1 and $\Diamond A \Rightarrow \Diamond I$. And in the light of the symmetry of *some*, [272]$A\Diamond A\Diamond I$-4 can be obtained by replacing the conclusion $\Diamond$*some(S, P)* in [046]$A\Diamond A\Diamond I$-1 with $\Diamond$*some(P, S)*. And then [318]$\Box EA\Box O$-4 can be derived by transforming [272]$A\Diamond A\Diamond I$-4 according to Propositional Deformation Rule 1, just as proved in Theorem 5 and Theorem 7. The other arguments are similar.

Theorem 28: Reducible relations between the following modal syllogisms:

(28.1) [011]□*E*□*A*□*E*-1 ⇒ [321]□*E*□*I*◇*O*-4

(28.2) [011]□*E*□*A*□*E*-1 ⇒ [322]□*E*◇*I*◇*O*-4

(28.3) [011]□*E*□*A*□*E*-1 ⇒ [323]□*EI*◇*O*-4

(28.4) [013]□*E*◇*A*◇*E*-1 ⇒ [324]□*E*□*I*□*O*-4

(28.5) [014]◇*E*□*A*◇*E*-1 ⇒ [325]◇*E*□*I*◇*O*-4

(28.6) [015]□*EA*□*E*-1 ⇒ [326]□*E*□*IO*-4

(28.7) [015]□*EA*□*E*-1 ⇒ [327]□*EIO*-4

(28.8) [016]*E*□*A*□*E*-1 ⇒ [328]*E*◇*I*◇*O*-4

(28.9) [016]*E*□*A*□*E*-1 ⇒ [329]□*E*◇*IO*-4

(28.10) [016]*E*□*A*□*E*-1 ⇒ [330]□*E*□*IO*-4

(28.11) [016]*E*□*A*□*E*-1 ⇒ [331]*EI*◇*O*-4

(28.12) [019]□*E*◇*AE*-1 ⇒ [332]□*EI*□*O*-4

(28.13) [020]*AA*◇*A*-1 ⇒ [333]*E*□*IO*-4

Proof: Similar to Theorem 18-21 and Theorem 27, Theorem 28 can be derived from the symmetry of *no*, Fact 2 and Propositional Deformation Rule 1. More specifically, for (28.10), it is clear that [031]*E*□*A*◇*E*-1 can be followed from [016]*E*□*A*□*E*-1 and □*E*⇒◇*E*. And [160]□*E*□*IO*-2 can be obtained from [031]*E*□*A*◇*E*-1 in terms of Propositional Deformation Rule 1, just as proved in Theorem 5 and Theorem 7. And then [330]□*E*□*IO*-4 can be followed by substituting □*some(M, S)* for the minor premise □*some(S, M)* in [160]□*E*□*IO*-2. The other verifications are similar or easier.

Theorem 29: Reducible relations between the following modal syllogisms:

(29.1) [011]□*E*□*A*□*E*-1 ⇒ [334]□*I*□*A*◇*I*-4

(29.2) [011]□*E*□*A*□*E*-1 ⇒ [335]◇*I*□*A*◇*I*-4

(29.3) [011]$\Box E \Box A \Box E$-1 $\Rightarrow$ [336]$\Box IA \Diamond I$-4

(29.4) [013]$\Box E \Diamond A \Diamond E$-1 $\Rightarrow$ [337]$\Box I \Diamond A \Diamond I$-4

(29.5) [014]$\Diamond E \Box A \Diamond E$-1 $\Rightarrow$ [338]$\Box I \Box A \Box I$-4

(29.6) [015]$\Box EA \Box E$-1 $\Rightarrow$ [339]$I \Diamond A \Diamond I$-4

(29.7) [015]$\Box EA \Box E$-1 $\Rightarrow$ [340]$IA \Diamond I$-4

(29.8) [015]$\Box EA \Box E$-1 $\Rightarrow$ [341]$I \Box A \Diamond I$-4

(29.9) [016]$E \Box A \Box E$-1 $\Rightarrow$ [342]$\Box I \Box AI$-4

(29.10) [016]$E \Box A \Box E$-1 $\Rightarrow$ [343]$\Box I \Diamond AI$-4

(29.11) [016]$E \Box A \Box E$-1 $\Rightarrow$ [344]$\Box IAI$-4

(29.12) [019]$\Box E \Diamond AE$-1 $\Rightarrow$ [345]$\Box IA \Diamond I$-4

(29.13) [020]$EA \Diamond E$-1 $\Rightarrow$ [346]$I \Box AI$-4

Proof: Similar to Theorem 18-21 and Theorem 27-28, Theorem 29 can be followed from the symmetry of *some*, Fact 2 and Propositional Deformation Rule 2. More specifically, for (29.8), it is clear that [029]$\Box EA \Diamond E$-1 can be obtained from [015]$\Box EA \Box E$-1 and $\Diamond E \Rightarrow \Diamond E$. Then [239]$I \Box A \Diamond I$-3 can be deducible from [029]$\Box EA \Diamond E$-1 according to Propositional Deformation Rule 2, just as proved in Theorem 4 and Theorem 6. And then with reference to the symmetry of *some*, [341]$I \Box A \Diamond I$-4 can be obtained by substituting $\Diamond$*some(P, M)* for the major premise $\Diamond$*some(M, P)* in [239]$I \Box A \Diamond I$-3. The other verifications follow the same pattern as the example just proved.

4. Conclusion and the Future Work

The above theorems show that the validity of the other 326 modal syllogisms can be deduced from the 20 basic valid modal syllogisms obtained by adding modal operators to the classical valid syllogisms *AAA*-1 and *EAE*-1. On the basis of the 20 valid modal syllogisms, this paper not only shows that the validity of the other 326 Aristotle's modal syllogisms can be derived by making full use of truth definition and symmetry of Aristotelian quantifiers in generalized quantifier theory, and propositional deformation rules in proof theory, but also shows that there are reducible relations between/among Aristotle's modal syllogisms. These innovative results are embodied in the 29 theorems proposed in this paper. It is hoped that these innovative achievements will make contributions to further research on Aristotle's and generalized modal syllogistic logic, and to promote knowledge representation and knowledge reasoning in computer science, and natural language information processing.

The study of the reducible relations between modal syllogisms is a necessary condition for studying the axiomatization of modal syllogistic. Up to now, the author has not yet found how to derive the other (386−326−20=) 38 valid modal syllogisms from the 20 basic valid modal syllogisms. If this problem is solved, then it is easier to axiomatize Aristotelian modal syllogistic.

It should be noted that the modal syllogisms studied in this paper are only Aristotle's modal syllogisms, because this paper only studies the modal syllogisms obtained by adding modal operators to the four Aristotelian quantifiers, *all*, *some*, *no*, and *not all*. Generalized modal syllogisms is obtained by adding modal operators to generalized quantifiers (such as *most*, *few*, *more...than*). But the research methods used in this paper provide a simple and reasonable mathematical model to study generalized modal syllogisms. How to use the methods to study the validity and reducible relations of generalized modal syllogisms? Due to limited time, it can only be left for future study.

References:

[1] A. Chagrov and M. Zakharyaschev. 1997, Modal Logic, Oxford: Clarendon Press.

[2] B. F. Chellas, 1980, Modal Logic: an Introduction, Cambridge University Press.

[3] P. T. Geach, 1964, "Review of McCall (1963)", Ratio, 6: 200-206.

[4] P. T. Geach, 1972, Logic Matters, Oxford: Blackwell, 1972.

[5] J. Łukasiewicz, 1957, Aristotle's Syllogistic: From the Standpoint of Modern Formal Logic, Oxford: Clarendon Press, second edition.

[6] F. Johnson, 1989, "Models for modal syllogisms", Notre Dame Journal of Formal Logic, 30: 271-284.

[7] F. Johnson, 1993, "Modal ecthesis", History and Philosophy of Logic, 14: 171-182.

[8] F. Johnson, 1995, “Extended gergonne syllogisms”, Journal of Philosophical Logic, 23: 553-567.

[9] F. Johnson, 2004, “Aristotle’s modal syllogisms”, Handbook of the History of Logic, 1: 247-307.

[10] M. Malink, 2006, “A reconstruction of Aristotle’s modal syllogistic”, History and Philosophy of Logic, 27: 95–141.

[11] M. Malink, 2013, Aristotle's Modal Syllogistic, Cambridge, MA: Harvard University Press.

[12] S. McCall, 1963, Aristotle’s Modal Syllogisms, Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam.

[13] R. Patterson, 1990, “Conversion principles and the basis of Aristotle’s modal logic”, History and Philosophy of Logic, 11: 151-172.

[14] S. Peters and D. Westerståhl, Quantifiers in Language and Logic, Oxford: Claredon Press, 2006.

[15] P. Thom, 1996, The Logic of Essentialism: An Interpretation of Aristotle’s Modal Syllogistic, (Synthese Historical Library 43), Dordrecht: Kluwer.

[16] S. K. Thomason, 1993, “Semantic analysis of the modal syllogistic”, Journal of Philosophical Logic, 22: 111–128.

[17] S. K. Thomason, 1997, “Relational models for the modal syllogistic”, Journal of Philosophical Logic, 26: 129–1141.

[18] J. van Benthem, 1984, “Questions about quantifiers”, Journal of Symbol Logic, 49(2): 443- 466.

[19] D. Westerståhl, 1989, “Aristotelian syllogisms and generalized quantifiers”, Studia Logica, XLVII(4): 577-585.

[20] X. Zhang, 2011， A Study of Properties of Generalized Quantifiers, Ph.D. dissertation, Chinese Academy of Social Sciences. (in Chinese)

[21] X. Zhang, 2014, A Study of Generalized Quantifier Theory, Xiamen University Press. (in Chinese)

[22] X. Zhang and S. Li, 2016, “Research on the formalization and axiomatization of classical syllogisms”, Journal of Hubei University (Philosophy and social sciences), 6: 32-37. (in Chinese)

[23] X. Zhang, Screening out all Valid Aristotelian Modal Syllogisms, to appear