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# INVESTIGATION OF A PROBLEM ON MOTION OF VISCOELASTIC MEDIUM UNDER THE ACTION OF REGID CYLINDER

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## ABSTRACT

Problems on motion of cylindrical bodies in continuum are widely represented in the monograph [1]. The problems represented in the reference refer to the motion of a cylinder in acoustic or elastic medium and are solved by operational method with finding originals by numericl methods [2].

Below we construct special solutions stipulated by possibility of obtaining analytic expressions that however may be used for construction of any solutions by numerical methods.

**Key words:** dilatation, deformation, cylinder, motion, displacement.

We'll consider motion of cylindrical rigid body in medium described by the Foight model for shear deformation and elastic dilatation.

Here instead of Hook's generalized law there will be:

$$\sigma_{ij} = \left(\lambda + \lambda' \frac{\partial}{\partial t}\right) \Delta \cdot \delta_{ij} + 2\left(\mu + \mu' \frac{\partial}{\partial t}\right) \varepsilon_{ij} \tag{1}$$

$$\sigma = k\Delta; \tag{2}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_j}{\partial x_j}\right) \tag{3}$$

where $\sigma_{ij}$ - stress tensor components,

$\varepsilon_{ij}$ - strain tensor components,

$\delta$ - hydrostatic stress,

$\Delta$ - volume strain (dilatation),

$\lambda_1$ - Liame densities,

$\lambda_1$ - viscosity coefficients,

$\Delta$ - Cronecker symbol,

K – volume compression coefficient,

$u_1$ – displacement vector projections,

$x_1$ - parameter of Cartesian coordinates.

From equations (1) we can write

$$\sigma_{11} + \sigma_{22} + \sigma_{33} = \xi\sigma = \left[3\left(\lambda + \lambda' \frac{\partial}{\partial t}\right) + 2\mu + 2\mu' \frac{\partial}{\partial t}\right] \cdot \Delta$$

Substituting into (2) we get

$$3\left(\lambda + \lambda' \frac{\partial}{\partial t}\right) + 2\mu + 2\mu' \frac{\partial}{\partial t} = 3k$$

Whence

$$k = \lambda + \frac{2}{3}\mu, \; \lambda' = -\frac{2}{3}\mu'.$$

Taking into account the last one in (1) we'll have

$$\sigma_{ij} = \left(\lambda - \frac{2}{3}\mu' \frac{\partial}{\partial t}\right) \cdot \Delta \cdot \delta_{ij} + 2\left(\mu + \mu' \frac{\partial}{\partial t}\right) \varepsilon_{ij} \tag{4}$$

Further we consider plane strain state. In this case motion equations have the form

$$\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} = \rho \frac{\partial^2 u_x}{\partial t^2}$$

$$\frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} = \rho \frac{\partial^2 u_y}{\partial t^2} \tag{5}$$

Strain law (3) will take the form

$$\sigma_{11} = \left(\lambda - \frac{2}{3}\mu' \frac{\partial}{\partial t}\right)(\varepsilon_{11} + \varepsilon_{22}) + 2\left(\mu + \mu' \frac{\partial}{\partial t}\right) \varepsilon_{11}$$

$$\sigma_{22} = \left(\lambda - \frac{2}{3}\mu' \frac{\partial}{\partial t}\right)(\varepsilon_{11} + \varepsilon_{22}) + 2\left(\mu + \mu' \frac{\partial}{\partial t}\right) \varepsilon_{22}$$

$$\sigma_{12} = 2\left(\mu + \mu' \frac{\partial}{\partial t}\right)\varepsilon_{12} \tag{6}$$

Substituting (6) into (5) with using (3) and accepting

$$u_x = \frac{\partial \varphi}{\partial x} - \frac{\partial \psi}{\partial y}; \quad u_y = \frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial y}$$

We get the following motion equations

$$\left(\lambda + 2\mu + \frac{4}{3}\mu' \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) = \rho \frac{\partial^2 \varphi}{\partial t^2} \tag{7'}$$

$$\left(\mu + \mu' \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = \rho \frac{\partial^2 \psi}{\partial t^2} \tag{8'}$$

Or in a polar system of coordinates

$$\left(\lambda + 2\mu + \frac{4}{3}\mu' \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial \theta^2}\right) = \rho \frac{\partial^2 \varphi}{\partial t^2} \tag{9}$$

$$\left(\mu + \mu' \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial \theta^2}\right) = \rho \frac{\partial^2 \psi}{\partial t^2} \tag{10}$$

But if dilatational and shear properties of the material are the same, then the strain law accepts the form [3]

$$\sigma_{ij} = (1 + L)\,(\lambda \Delta \cdot \delta_{ij} + 2\mu \varepsilon_{ij}) \tag{11}$$

where L – is a linear time operator and motion equations are obtained replacing the Liame constants by appropriate operators.

Displacements in polar coordinates are expressed as

$$u = \frac{\partial \varphi}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial \theta} \; ; \; v = \frac{\partial \psi}{\partial r} + \frac{1}{r}\frac{\partial \varphi}{\partial \theta} \qquad (12)$$

where u and v – are radial and peripheral constituents of displacement vector, v – is distance from the pole, θ - is a polar angle.

We look for the solution of equations (7) and (8) in the form $\varphi = \varphi_i \cos\theta$ and $\psi = \psi_i \sin\theta$, and displacements $u = u_i \cos\theta$ and $v = v_i \sin\theta$.

Having substituted in (7) , (8) and (10) we get

$$\left(\lambda + 2\mu + \nu_1 \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 \varphi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi_1}{\partial r} - \varphi_1\right) = \rho \frac{\partial^2 \varphi_1}{\partial t^2} \qquad (13)$$

$$\left(\mu + \nu_2 \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 \psi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_1}{\partial r} - \psi_1\right) = \rho \frac{\partial^2 \psi_1}{\partial t^2} \qquad (14)$$

where

$$\nu_1 = \frac{4}{3}\mu' \text{ и } \nu_2 = \mu' .$$

$$u_1 = \frac{\partial \varphi_1}{\partial r} - \frac{\psi_1}{r} \; ; \; v_1 = \frac{\partial \psi_1}{\partial r} - \frac{\varphi_1}{r} \qquad (15)$$

We solve equations (11) and (12) for initial rest state of medium and condition of motion without giving up the cylinder.

Applying the Laplace-Carson transformation to (11) and (120 we have the following solutions bounded at infinity.

$$\bar{\varphi}_1 = C K_1\left(\frac{pr}{\sqrt{a^2 + \nu_1 p}}\right) \; ; \; \bar{\psi}_1 = D K_1\left(\frac{pr}{\sqrt{a^2 + \nu_2 p}}\right) \qquad (16)$$

where p – is a transformation parameter

$$a = \sqrt{\frac{\lambda + 2\mu}{\rho}} \ ; \ b = \sqrt{\frac{\mu}{\rho}} .$$

At the point of adjointment of medium to the cylinder surface the displacement of cylinder will be equal to displacement $\varepsilon$ of the cylinder, if we take direction of $\varepsilon$ along the axis X, then

$$\xi = u\cos\theta + v\sin\theta = \left(\frac{\partial \varphi_1}{\partial r} - \frac{\psi_1}{r}\right)\cos^2\theta + \left(\frac{\partial \psi_1}{\partial r} - \frac{\varphi_1}{r}\right)\sin^2\theta$$

As the cylinder is rigid, then all its points have (in the case of translational motion) the same displacement, don't depend on $\theta$ and consequently

$$\frac{\partial \varphi_1}{\partial r} - \frac{\psi_1}{r} = \frac{\partial \psi_1}{\partial r} - \frac{\varphi_1}{r} \tag{17}$$

Having substituted the solution (14), equated $v = v_0$ in (15) preliminarily having written it in images allowing for

$$\frac{\partial \overline{\varphi}_1}{\partial r} = -C\left[\frac{pK_0\left(\frac{pr_0}{\sqrt{a^2 + \nu_1 p}}\right)}{\sqrt{a^2 + \nu_1 p}} + \frac{K_1\left(\frac{pr_0}{\sqrt{a^2 + \nu_1 p}}\right)}{r_0}\right]$$

$$\frac{\partial \overline{\psi}_1}{\partial r} = -D\left[\frac{pK_0\left(\frac{pr}{\sqrt{b^2 + \nu_2 p}}\right)}{\sqrt{b^2 + \nu_2 p}} + \frac{K_1\left(\frac{pr_0}{\sqrt{b^2 + \nu_2 p}}\right)}{r_0}\right] \tag{18}$$

We get proportion

$$\frac{C\sqrt{b^2 + \nu_2 p}}{K_0\left(\frac{pr_0}{\sqrt{b^2 + \nu_2 p}}\right)} = \frac{D\sqrt{a^2 + \nu_1 p}}{K_0\left(\frac{pr_0}{\sqrt{a^2 + \nu_1 p}}\right)} = \frac{F}{\sqrt{(a^2 + \nu_1 p)(b^2 + \nu_2 p)}} \text{ where}$$

$$C = \frac{F K_0\left(\frac{p r_0}{\sqrt{b^2 + \nu_2 p}}\right)}{\sqrt{(a^2 + \nu_1 p)}\,(b^2 + \nu_2 p)}\,;\quad D = \frac{F K_0\left(\frac{p r_0}{\sqrt{(a^2 + \nu_1 p)}}\right)}{(a^2 + \nu_1 p)\,\sqrt{(b^2 + \nu_2 p)}} \tag{19}$$

Now, instead of two arbitrary constants C and D of solutions of equations (11) and 912) we have one F for displacements. Having substituted the expressions (16) and (17) into (13) we get

$$\bar{u}_1 = -\left[\frac{p K_0\left(\frac{p r_0}{\sqrt{b^2 + \nu_2 p}}\right)\cdot K_0\left(\frac{p r}{\sqrt{a^2 + \nu_1 p}}\right)}{(a^2 + \nu_1 p)(b^2 + \nu_2 p)} + \frac{K_0\left(\frac{p r_0}{\sqrt{b^2 + \nu_2 p}}\right)\cdot K_1\left(\frac{p r}{\sqrt{a^2 + \nu_1 p}}\right)}{r\,(b^2 + \nu_2 p)\sqrt{a^2 + \nu_1 p}}\right. +$$

$$\left. + \frac{K_0\left(\frac{p r_0}{\sqrt{a^2 + \nu_1 p}}\right)\cdot K_1\left(\frac{p r}{\sqrt{b^2 + \nu_2 p}}\right)}{r(a^2 + \nu_1 p)\sqrt{b^2 + \nu_2 p}}\right] F \tag{20}$$

$$\bar{v}_1 = -\left[\frac{p K_0\left(\frac{p r_0}{\sqrt{a^2 + \nu_1 p}}\right)\cdot K_0\left(\frac{p r}{\sqrt{b^2 + \nu_2 p}}\right)}{(a^2 + \nu_1 p)(b^2 + \nu_2 p)} + \frac{K_0\left(\frac{p r_0}{\sqrt{a^2 + \nu_1 p}}\right)\cdot K_1\left(\frac{p r}{\sqrt{b^2 + \nu_2 p}}\right)}{r\,(a^2 + \nu_1 p)\sqrt{b^2 + \nu_2 p}}\right. +$$

$$\left. + \frac{K_0\left(\frac{p r_0}{\sqrt{b^2 + \nu_2 p}}\right)\cdot K_1\left(\frac{p r}{\sqrt{a^2 + \nu_1 p}}\right)}{(b^2 + \nu_2 p)\sqrt{a^2 + \nu_1 p}}\right] F$$

Then we find originals of solutions obtained in images.

For that we use [4] the originals of the function

$$pK_0(\alpha\sqrt{p}) \to \frac{\exp\left(-\frac{\alpha^2}{4t}\right)}{2t}$$

Using the displacement theorem we pass from P to P-2

$$pK_0(\alpha\sqrt{p-2}) \to \frac{\exp\left(2t-\frac{\alpha^2}{4t}\right)}{2t}.$$

Using the formula of passage from p to $p+\frac{1}{p}$, we get

$$K_0\left(\alpha\frac{p-1}{\sqrt{p}}\right) \to \frac{1}{2}\int_0^t J_0(2\sqrt{(t-\tau)\tau})\, e^{2\tau-\frac{\alpha^2}{4\tau}}\frac{d\tau}{\tau}$$

Passing from p to p+1 we get

$$\frac{p}{p+1}K_0\left(\alpha\frac{p}{\sqrt{p+1}}\right) \to \frac{1}{2}e^{-t}\int_0^t J_0(2\sqrt{(t-\tau)\tau})\, e^{2\tau-\frac{\alpha^2}{4\tau}}\frac{d\tau}{\tau}$$

Introducing physical constants by the formula

$$\gamma = \frac{\nu}{a^2}\ ;\ \alpha = \frac{ra}{\nu},$$

we finally get $\frac{v_1 p_1}{a^2+v_1}$

By the similarity theorem

$$\frac{\gamma p}{\gamma p+1}K_0\left(\frac{\alpha\gamma p}{\sqrt{\gamma p+1}}\right) \to \frac{1}{2}e^{-\frac{t}{\gamma}}\int_0^t J_0\left(\frac{2}{\gamma}\sqrt{(t-\tau)\tau}\right) e^{\frac{2\tau}{\gamma}-\frac{\alpha^2\gamma}{4\tau}}\frac{d\tau}{\tau}$$

Similarly

$$\frac{\nu_2 p}{b^2+\nu_2 p} K_0\left(\frac{pr}{\sqrt{b^2+\nu_2 p}}\right) \to \frac{1}{2} e^{-\frac{b^2 t}{\nu_2}} \int_0^t J_0\left(\frac{2b^2}{\nu_2}\sqrt{(t-\tau)\tau}\right) e^{\frac{2b^2\tau}{\nu_2}-\frac{r^2}{4\nu_2\tau}} \frac{d\tau}{\tau} \quad (21)$$

Using the original of the function [4]

$$\sqrt{p} K_1\left(2\sqrt{\alpha p}\right) \to \frac{\exp\left(-\frac{\alpha}{t}\right)}{2\sqrt{\alpha}}$$

And performing passages similar to previous ones we get

$$\frac{a}{\sqrt{a^2+\nu_1 p}} K_1\left(\frac{pr}{\sqrt{a^2+\nu_1 p}}\right) \to \frac{a}{r} e^{-\frac{a^2 t}{\nu_1}} \int_0^t J_0\left(\frac{2a^2}{\nu_1}\sqrt{(t-\tau)\tau}\right) e^{\frac{2a^2\tau}{\nu_1}-\frac{r^2}{4\nu_1\tau}} d\tau \quad (22)$$

$$\frac{b}{\sqrt{b^2+\nu_2 p}} K_1\left(\frac{pr}{\sqrt{b^2+\nu_2 p}}\right) \to \frac{b}{r} e^{-\frac{a^2 t}{\nu_2}} \int_0^t J_0\left(\frac{2a^2}{\nu_1}\sqrt{(t-\tau)\tau}\right) e^{\frac{2a^2\tau}{\nu_1}-\frac{r^2}{4\nu_1\tau}} d\tau \quad (23)$$

Using the originals from (19), (20) and (22) in (18) we get an expression of displacements in originals

$$\frac{u_1}{F_0}=\frac{1}{4\nu_1\nu_2}e^{-\frac{b^2 t}{\nu_2}}\int_0^t e^{\left(\frac{b^2}{\nu_2}-\frac{a^2}{\nu_1}\right)\eta}\left(\int_0^\eta J_0\left(\frac{2a^2}{\nu_1}\sqrt{(\eta-\tau)\tau}\right)e^{\frac{1}{\nu_1}\left(2a^2\tau-\frac{r_0^2}{4\tau}\right)}\frac{d\tau}{\tau}\right)\times$$

$$\times\left(\int_0^{t-\eta}J_0\left(\frac{2b^2}{\nu_2}\sqrt{(t-\eta-\tau)\tau}\right)e^{\frac{1}{\nu_2}\left(2a^2\tau-\frac{r_0^2}{4\tau}\right)}\frac{d\tau}{\tau}\right)d\eta-\frac{1}{2\nu_2 r}e^{-\frac{a^2 t}{\nu_1}}\int_0^t e^{\left(\frac{a^2}{\nu_1}-\frac{b^2}{\nu_2}\right)\eta}\times$$

$$\times\left(\int_0^\eta J_0\left(\frac{2b^2}{\nu_2}\sqrt{(\eta-\tau)\tau}\right)e^{\frac{1}{\nu_2}\left(2b^2\tau-\frac{r_0^2}{4\tau}\right)}\frac{d\tau}{\tau}\right)\left(\int_0^{t-\eta}J_0\left(\frac{2a^2}{\nu_1}\sqrt{(t-\eta-\tau)\tau}\right)e^{\frac{1}{\nu_1}\left(2a^2\tau-\frac{r_0^2}{4\tau}\right)}d\tau\right)d\eta-$$

$$-\frac{1}{2\nu_2 r}e^{-\frac{b^2 t}{\nu_1}}\int_0^t e^{\left(\frac{b^2}{\nu_2}-\frac{a^2}{\nu_1}\right)\eta}\left(\int_0^\eta J_0\left(\frac{2a^2}{\nu_1}\sqrt{(\eta-\tau)\tau}\right)e^{\frac{1}{\nu_1}\left(2b^2\tau-\frac{r_0^2}{4\tau}\right)}\frac{d\tau}{\tau}\right)\times$$

$$\times\left(\int_0^{t-\eta}J_0\left(\frac{2b^2}{\nu_2}\sqrt{(t-\eta-\tau)\tau}\right)e^{\frac{1}{\nu_2}\left(2b^2\tau-\frac{r_0^2}{4\tau}\right)}d\tau\right)d\eta$$

We can get an expression for $v_1$, in a similar way.

Here the multiplier F in (18) is taken equal to constant number.

The last expression may be considered as a special solution, since the function F may be included into boundary condition in view of its independence from v.

At the boundary $r=r_0$ this solution tends to zero as $\tau\to 0$, sinse $[r_0/(at)]\to 0$.

Thus, we have a solution strongly growing beginning with zero with the subsequent delay of growth.

Having the given boundary condidition and the obtained special solution we construct the solution by numerical methods worked out in the papers [1] and [2].

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